Digital Image Processing

Lecture 7. Filtering in the Frequency Domain

Fall 2010

Outline

- Fourier Transform
- Filtering in Fourier Transform Domain

Fourier Series and Fourier Transform: History

- *Jean Baptiste Joseph Fourier*, French mathematician and physicist (03/21/1768-05/16/1830) http://en.wikipedia.org/wiki/Joseph_Fourier

Orphaned: at nine

Egyptian expedition with **Napoleon I**: 1798
Governor of Lower Egypt

Permanent Secretary of the French Academy of Sciences: 1822

Théorie analytique de la chaleur : 1822

(**The Analytic Theory of Heat**)

Fourier Series and Fourier Transform: History

- Fourier Series

  Any periodic function can be expressed as the sum of sines and /or cosines of different frequencies, each multiplied by a different coefficients

- Fourier Transform

  Any function that is not periodic can be expressed as the integral of sines and /or cosines multiplied by a weighing function

Fourier Series: Example

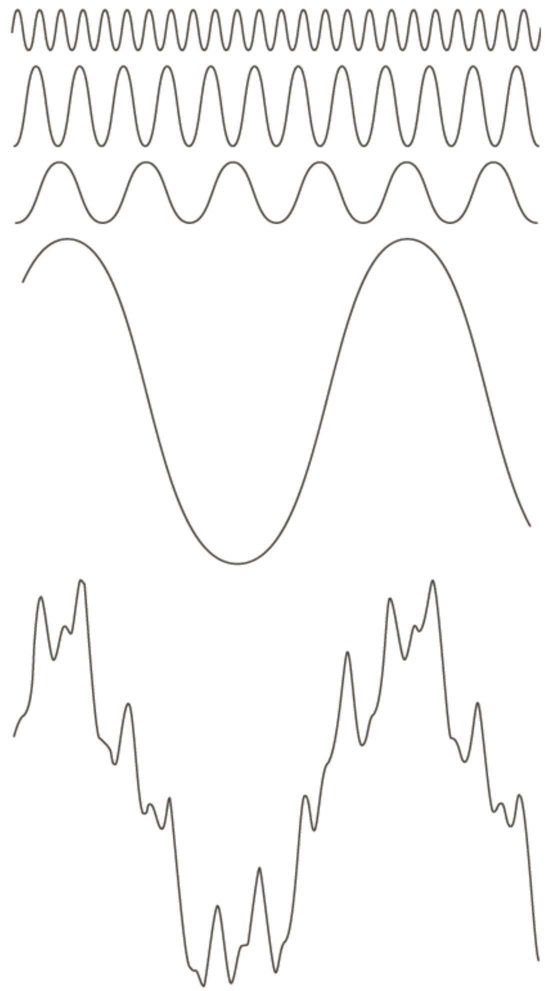


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

Preliminary Concepts

$j = \sqrt{-1}$, a complex number

$$C = R + jI$$

the conjugate

$$C^* = R - jI$$

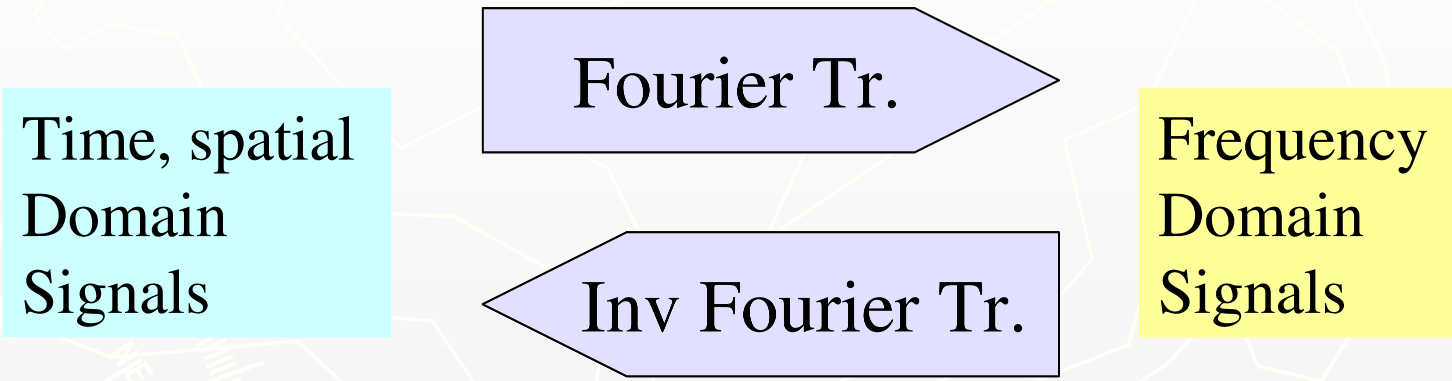
$|C| = \sqrt{R^2 + I^2}$ and $\theta = \arctan(I / R)$

$$C = |C|(\cos\theta + j\sin\theta)$$

Using Euler's formula,

$$C = |C| e^{j\theta}$$

Fourier Tr. and Frequency Domain



Fourier Transform: One Continuous Variable

The *Fourier Transform* of a continous function $f(t)$

$$F(\mu) = \Im\{f(t)\} = \int_{-\infty}^{\infty} f(t)e^{-j2\pi\mu t}dt$$

The *Inverse Fourier Transform* of $F(\mu)$

$$f(t) = \Im^{-1}\{F(\mu)\} = \int_{-\infty}^{\infty} F(\mu)e^{j2\pi\mu t}d\mu$$

Fourier Transform: One Continuous Variable

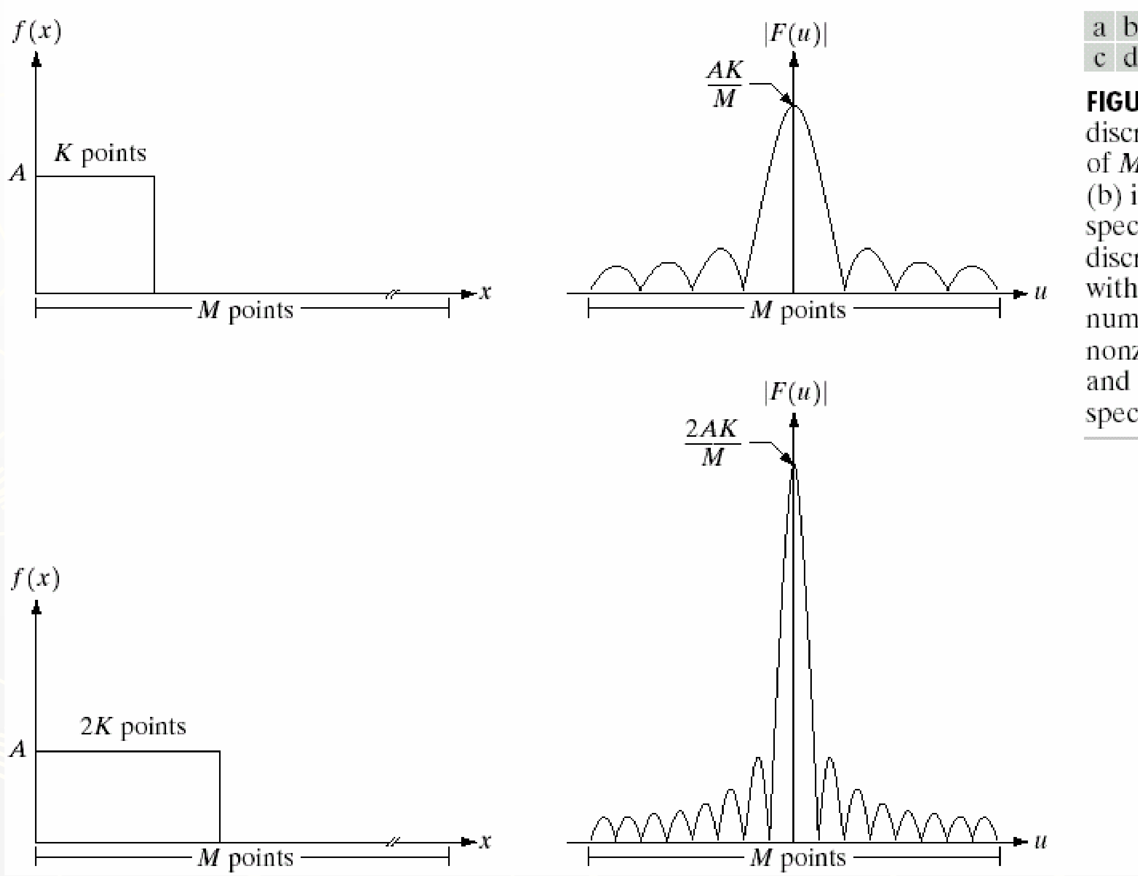


FIGURE 4.2 (a) A discrete function of M points, and (b) its Fourier spectrum. (c) A discrete function with twice the number of nonzero points, and (d) its Fourier spectrum.

Fourier Tr. and Frequency Domain (cont.)

1-D, Discrete case

Fourier Tr.: $$F(u) = \frac{1}{M}\sum_{x=0}^{M-1} f(x) e^{-j2\pi ux/M} \qquad u = 0,\ldots,M-1$$

Inv. Fourier Tr.: $$f(x) = \sum_{u=0}^{M-1} F(u) e^{j2\pi ux/M} \qquad x = 0,\ldots,M-1$$

$F(u)$ can be written as

$$F(u) = R(u) + jI(u) \quad \text{or} \quad F(u) = |F(u)| e^{-j\phi(u)}$$

where

$$|F(u)| = \sqrt{R(u)^2 + I(u)^2} \qquad \phi(u) = \tan^{-1}\left(\frac{I(u)}{R(u)}\right)$$

Relation Between Δx and Δu

For a signal $f(x)$ with M points, let spatial resolution Δx be space between samples in $f(x)$ and let frequency resolution Δu be space between frequencies components in $F(u)$, we have

$$\Delta u = \frac{1}{M\Delta x}$$

Example: for a signal f(x) with sampling period 0.5 sec, 100 point, we will get frequency resolution equal to

$$\Delta u = \frac{1}{100 \times 0.5} = 0.02 \text{ Hz}$$

This means that in $F(u)$ we can distinguish 2 frequencies that are apart by 0.02 Hertz or more.

2-D Discrete Fourier Transform and Its Inverse

DFT:

$$F(\mu,\nu) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)e^{-j2\pi(\mu x/M+\nu y/N)}$$

$\mu = 0,1,2,\ldots,M-1; \nu = 0,1,2,\ldots,N-1;$

$f(x,y)$ is a digital image of size $\mathrm{M}\times\mathrm{N}$.

IDFT:

$$f(x,y) = \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} F(\mu,\nu)e^{j2\pi(\mu x/M+\nu y/N)}$$

Properties of the 2-D DFT
Fourier Spectrum and Phase Angle

2-D DFT in polar form

$$F(u,v) = |F(u,v)|\,e^{j\phi(u,v)}$$

Fourier spectrum

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v)\right]^{1/2}$$

Power spectrum

$$P(u,v) = |F(u,v)|^2 = R^2(u,v) + I^2(u,v)$$

Phase angle

$$\phi(\mathrm{u,v}) = \arctan\left[\frac{I(u,v)}{R(u,v)}\right]$$

Properties of the 2-D DFT
relationships between spatial and frequency intervals

Let ΔT and ΔZ denote the separations between samples, then the seperations between the corresponding discrete, frequency domain variables are given by

$$\Delta\mu = \frac{1}{M\Delta T}$$

and $$\Delta\nu = \frac{1}{N\Delta Z}$$

Summary

Name	Expression(s)
1) Discrete Fourier transform (DFT) of $f(x, y)$	$F(u, v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) e^{-j2\pi(ux/M + vy/N)}$
2) Inverse discrete Fourier transform (IDFT) of $F(u, v)$	$f(x, y) = \frac{1}{MN} \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u, v) e^{j2\pi(ux/M + vy/N)}$
3) Polar representation	$F(u, v) = \lvert F(u, v) \rvert e^{j\phi(u,v)}$
4) Spectrum	$\lvert F(u, v) \rvert = \left[R^2(u, v) + I^2(u, v) \right]^{1/2}$ $R = \text{Real}(F); \quad I = \text{Imag}(F)$
5) Phase angle	$\phi(u, v) = \tan^{-1} \left[\frac{I(u, v)}{R(u, v)} \right]$
6) Power spectrum	$P(u, v) = \lvert F(u, v) \rvert^2$
7) Average value	$\overline{f}(x, y) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y) = \frac{1}{MN} F(0, 0)$

(*Continued*)

Summary

Name	Expression(s)
8) Periodicity (k_1 and k_2 are integers)	$F(u, v) = F(u + k_1M, v) = F(u, v + k_2N) = F(u + k_1M, v + k_2N)$ $f(x, y) = f(x + k_1M, y) = f(x, y + k_2N) = f(x + k_1M, y + k_2N)$
9) Convolution	$f(x, y) \bigstar h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m, n)h(x - m, y - n)$
10) Correlation	$f(x, y) \star h(x, y) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f^*(m, n)h(x + m, y + n)$
11) Separability	The 2-D DFT can be computed by computing 1-D DFT transforms along the rows (columns) of the image, followed by 1-D transforms along the columns (rows) of the result. See Section 4.11.1.
12) Obtaining the inverse Fourier transform using a forward transform algorithm.	$MNf^*(x, y) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F^*(u, v)e^{-j2\pi(ux/M + vy/N)}$ This equation indicates that inputting $F^*(u, v)$ into an algorithm that computes the forward transform (right side of above equation) yields $MNf^*(x, y)$. Taking the complex conjugate and dividing by MN gives the desired inverse. See Section 4.11.2.

Summary

Name	DFT Pairs
1) Symmetry properties	See Table 4.1
2) Linearity	$af_1(x, y) + bf_2(x, y) \Leftrightarrow aF_1(u, v) + bF_2(u, v)$
3) Translation (general)	$f(x, y)e^{j2\pi(u_0x/M + v_0y/N)} \Leftrightarrow F(u - u_0, v - v_0)$ $f(x - x_0, y - y_0) \Leftrightarrow F(u, v)e^{-j2\pi(ux_0/M + vy_0/N)}$
4) Translation to center of the frequency rectangle, $(M/2, N/2)$	$f(x, y)(-1)^{x+y} \Leftrightarrow F(u - M/2, v - N/2)$ $f(x - M/2, y - N/2) \Leftrightarrow F(u, v)(-1)^{u+v}$
5) Rotation	$f(r, \theta + \theta_0) \Leftrightarrow F(\omega, \varphi + \theta_0)$ $x = r\cos\theta \quad y = r\sin\theta \quad u = \omega\cos\varphi \quad v = \omega\sin\varphi$
6) Convolution theorem†	$f(x, y) \star h(x, y) \Leftrightarrow F(u, v)H(u, v)$ $f(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$

(*Continued*)

Summary

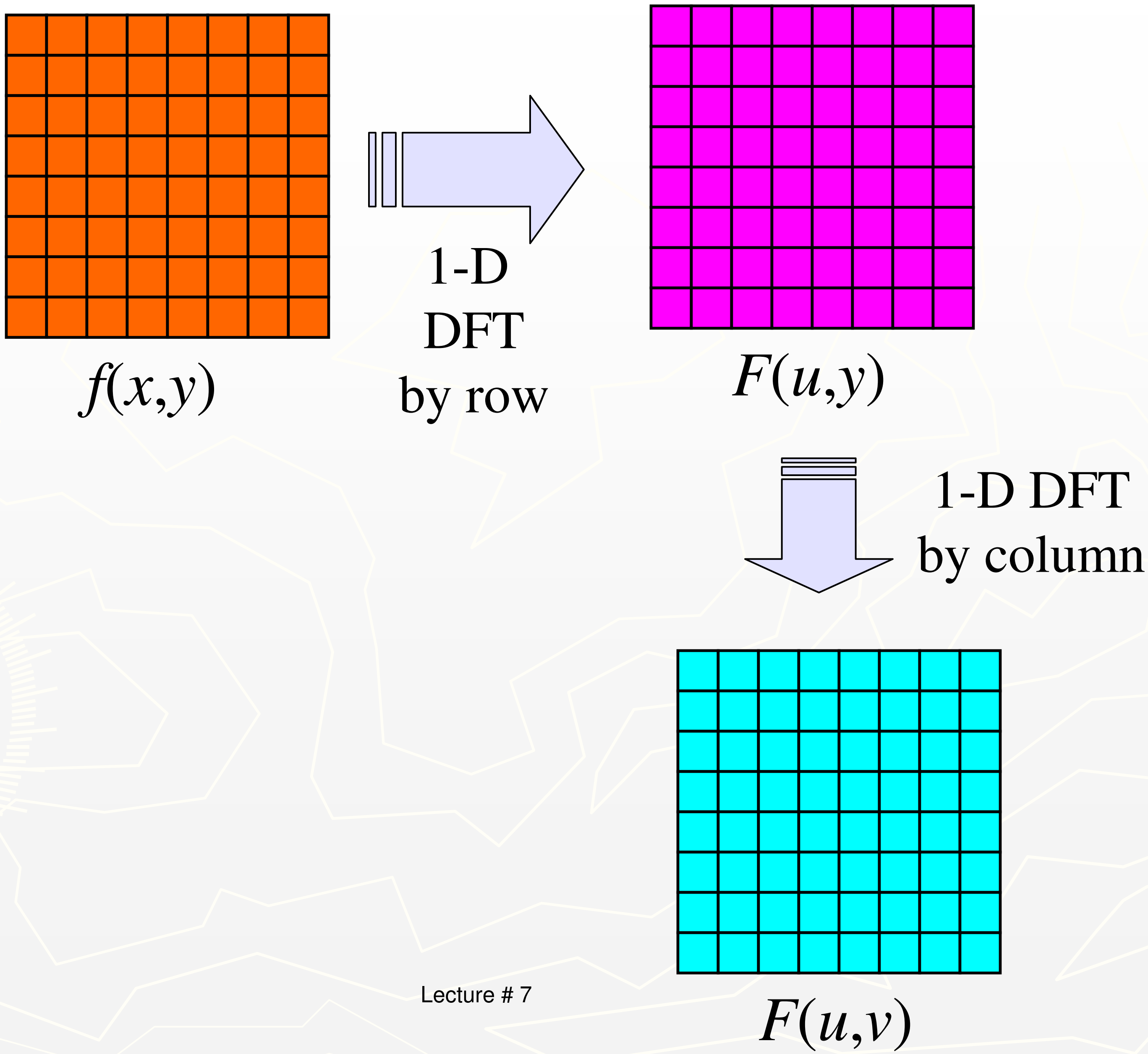
Name	DFT Pairs
7) Correlation theorem[†]	$f(x, y) \star h(x, y) \Leftrightarrow F^*(u, v)\, H(u, v)$ $f^*(x, y)h(x, y) \Leftrightarrow F(u, v) \star H(u, v)$
8) Discrete unit impulse	$\delta(x, y) \Leftrightarrow 1$
9) Rectangle	$\text{rect}[a, b] \Leftrightarrow ab \frac{\sin(\pi ua)}{(\pi ua)} \frac{\sin(\pi vb)}{(\pi vb)} e^{-j\pi(ua+vb)}$
10) Sine	$\sin(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $j\frac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) - \delta(u - Mu_0, v - Nv_0)\right]$
11) Cosine	$\cos(2\pi u_0 x + 2\pi v_0 y) \Leftrightarrow$ $\frac{1}{2}\left[\delta(u + Mu_0, v + Nv_0) + \delta(u - Mu_0, v - Nv_0)\right]$

The following Fourier transform pairs are derivable only for continuous variables, denoted as before by t and z for spatial variables and by μ and ν for frequency variables. These results can be used for DFT work by sampling the continuous forms.

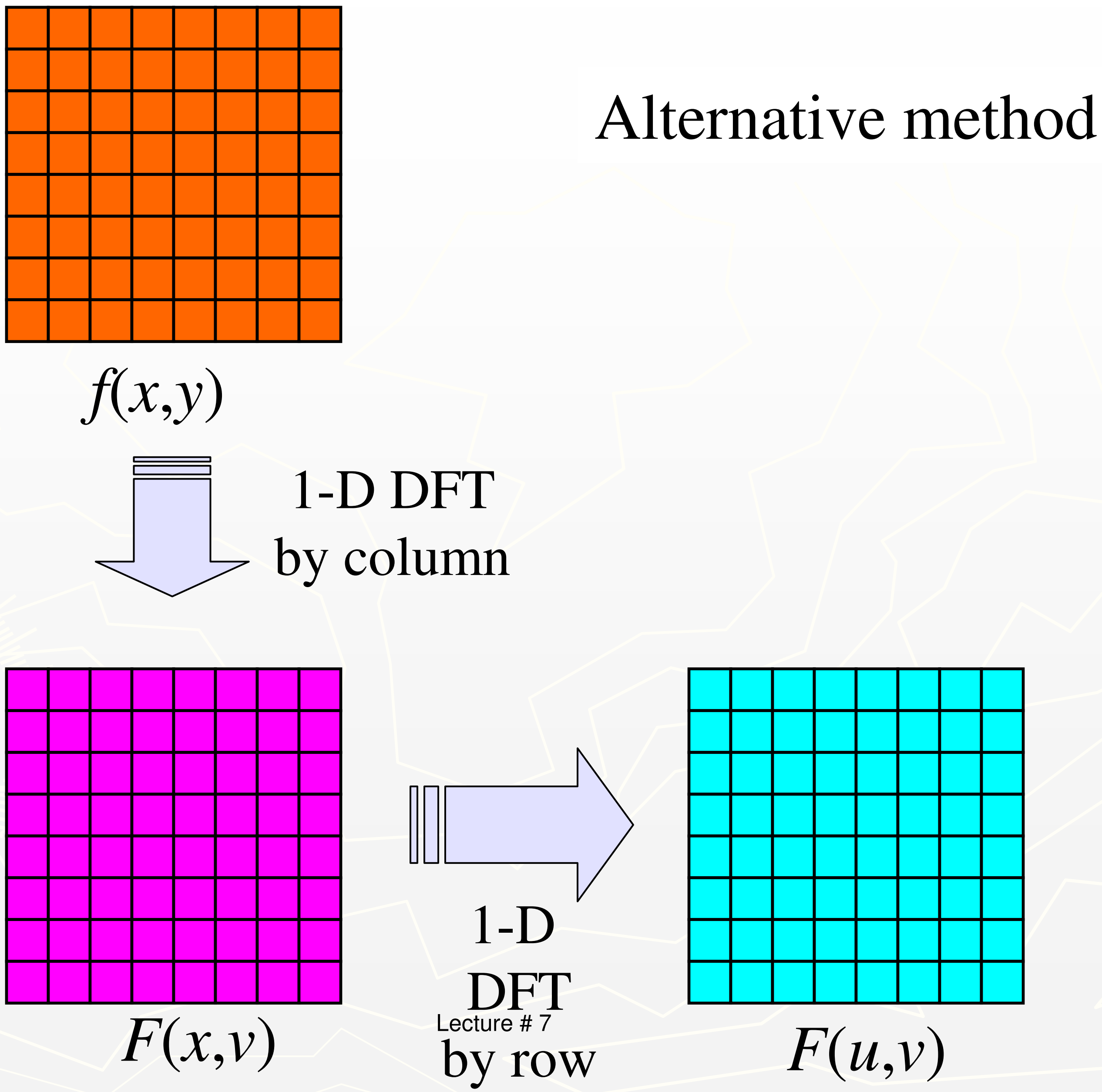
12) *Differentiation* (The expressions on the right assume that $f(\pm\infty, \pm\infty) = 0$.)	$\left(\frac{\partial}{\partial t}\right)^m \left(\frac{\partial}{\partial z}\right)^n f(t, z) \Leftrightarrow (j2\pi\mu)^m (j2\pi\nu)^n F(\mu, \nu)$ $\frac{\partial^m f(t, z)}{\partial t^m} \Leftrightarrow (j2\pi\mu)^m F(\mu, \nu); \frac{\partial^n f(t, z)}{\partial z^n} \Leftrightarrow (j2\pi\nu)^n F(\mu, \nu)$
13) *Gaussian*	$A2\pi\sigma^2 e^{-2\pi^2\sigma^2(t^2+z^2)} \Leftrightarrow Ae^{-(\mu^2+\nu^2)/2\sigma^2}$ (A is a constant)

[†]Assumes that the functions have been extended by zero padding. Convolution and correlation are associative, commutative, and distributive.

How to Perform 2-D DFT by Using 1-D DFT

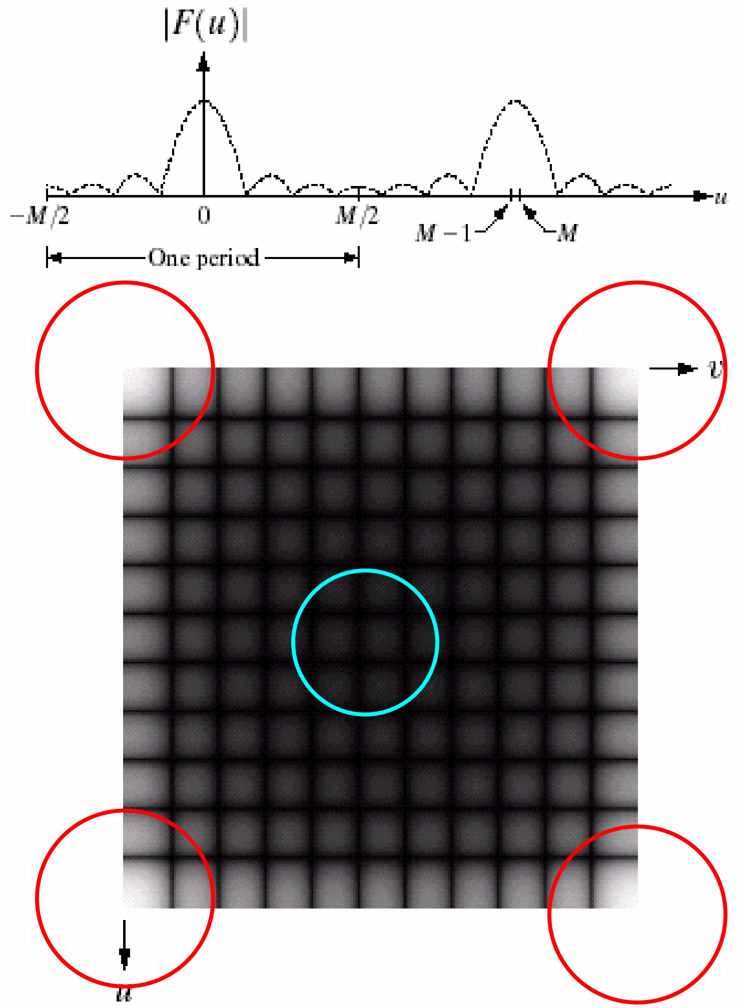


How to Perform 2-D DFT by Using 1-D DFT



Conventional Display for 2-D DFT

$F(u,v)$ has low frequency areas at corners of the image while high frequency areas are at the center of the image which is inconvenient to interpret.

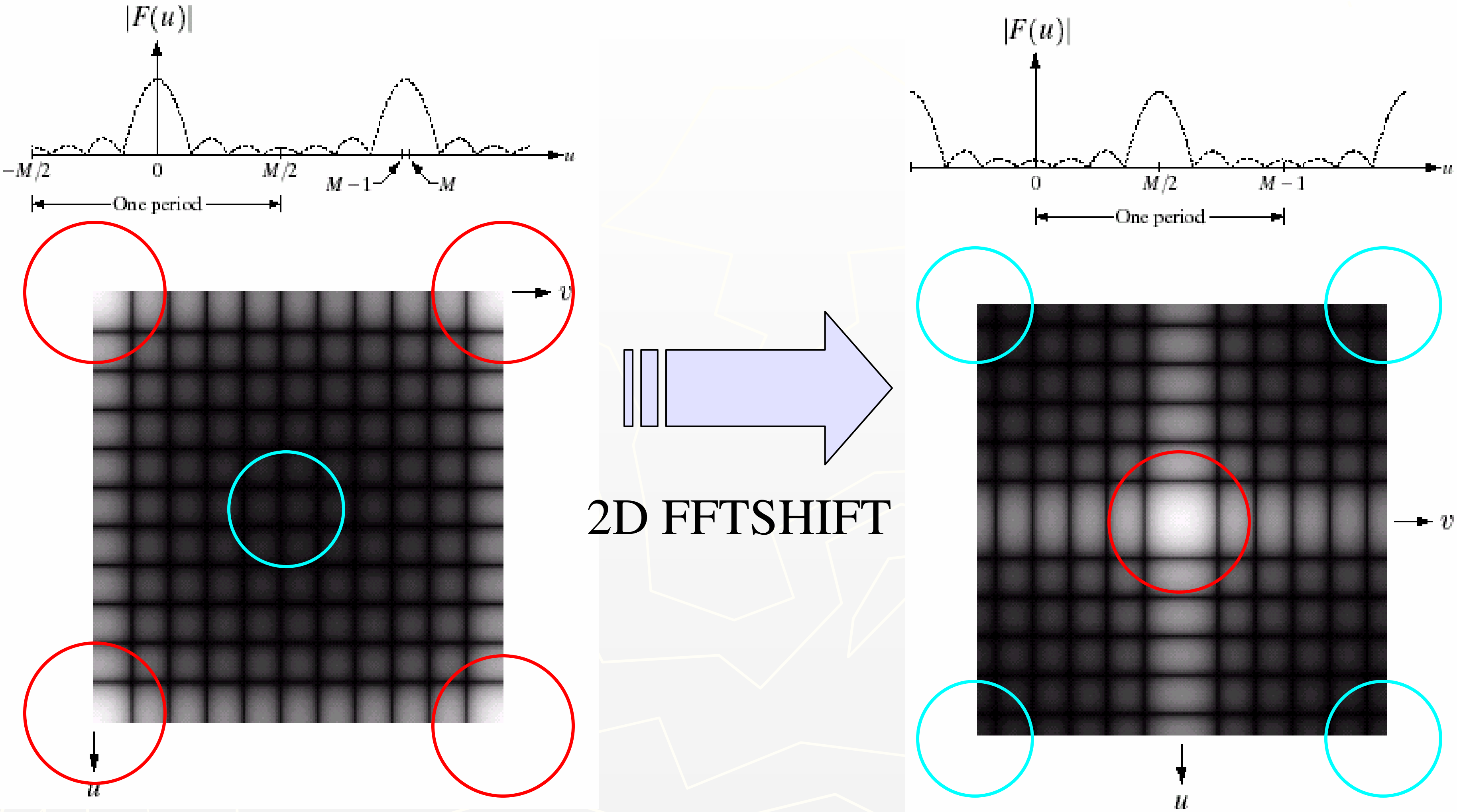


High frequency area

Low frequency area

2-D FFT Shift : Better Display of 2-D DFT

2-D FFT Shift is a MATLAB function: Shift the zero frequency of $F(u,v)$ to the center of an image.



High frequency area

Low frequency area

Shifting the Origin to the Center

If we put $u_0 = M/2$, and $v_0 = N/2$,

$$e^{j2\pi(u_0 x/M + v_0 x/N)} = e^{j\pi(x+y)}$$

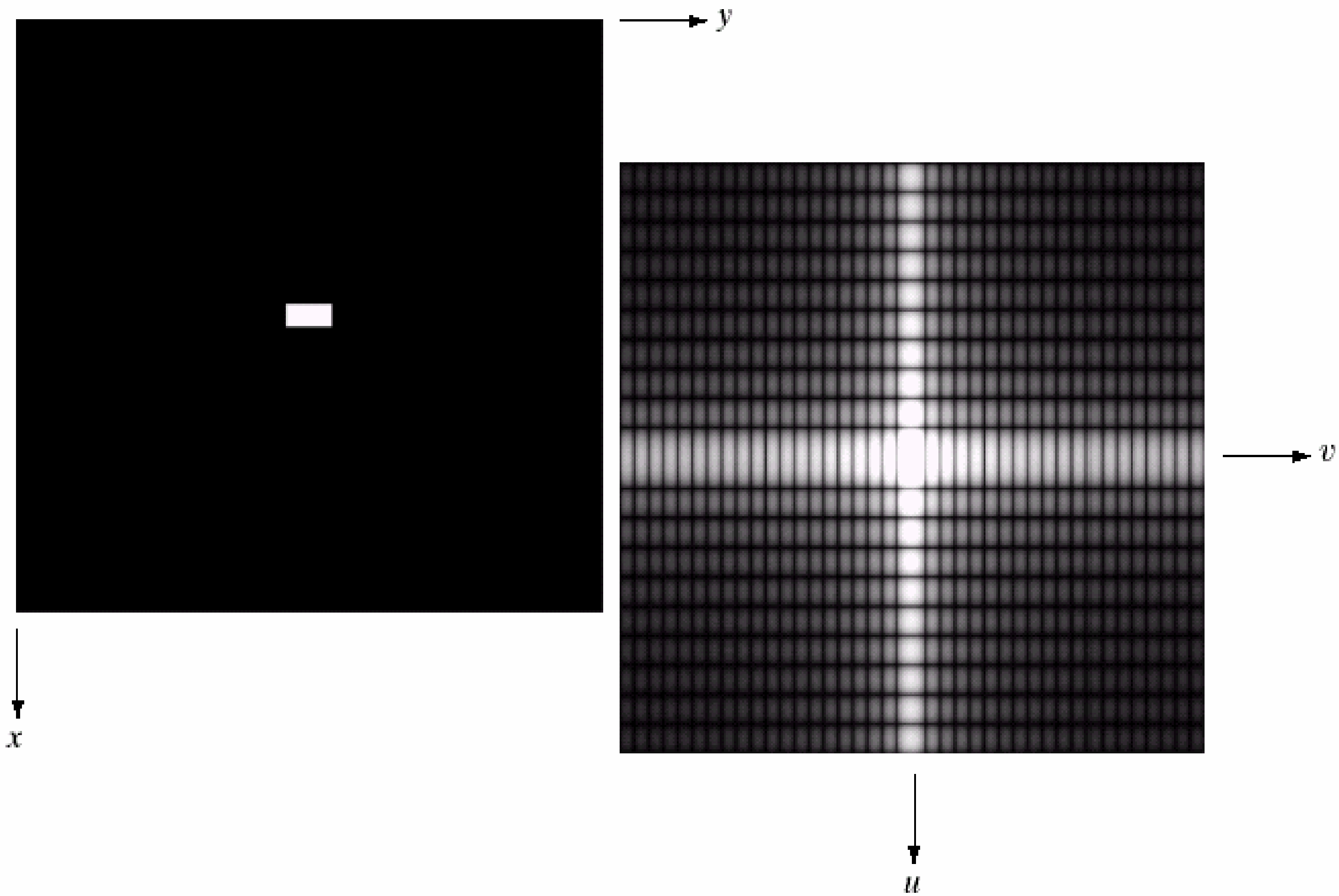
$$= (-1)^{(x+y)}$$

This means that

$$f(x,y)(-1)^{(x+y)} \Leftrightarrow F(u - M/2, v - N/2)$$

To transform the origin to the center in the transformed image the input is always multiplied by the factor $(-1)^{(x+y)}$

Example of 2-D DFT

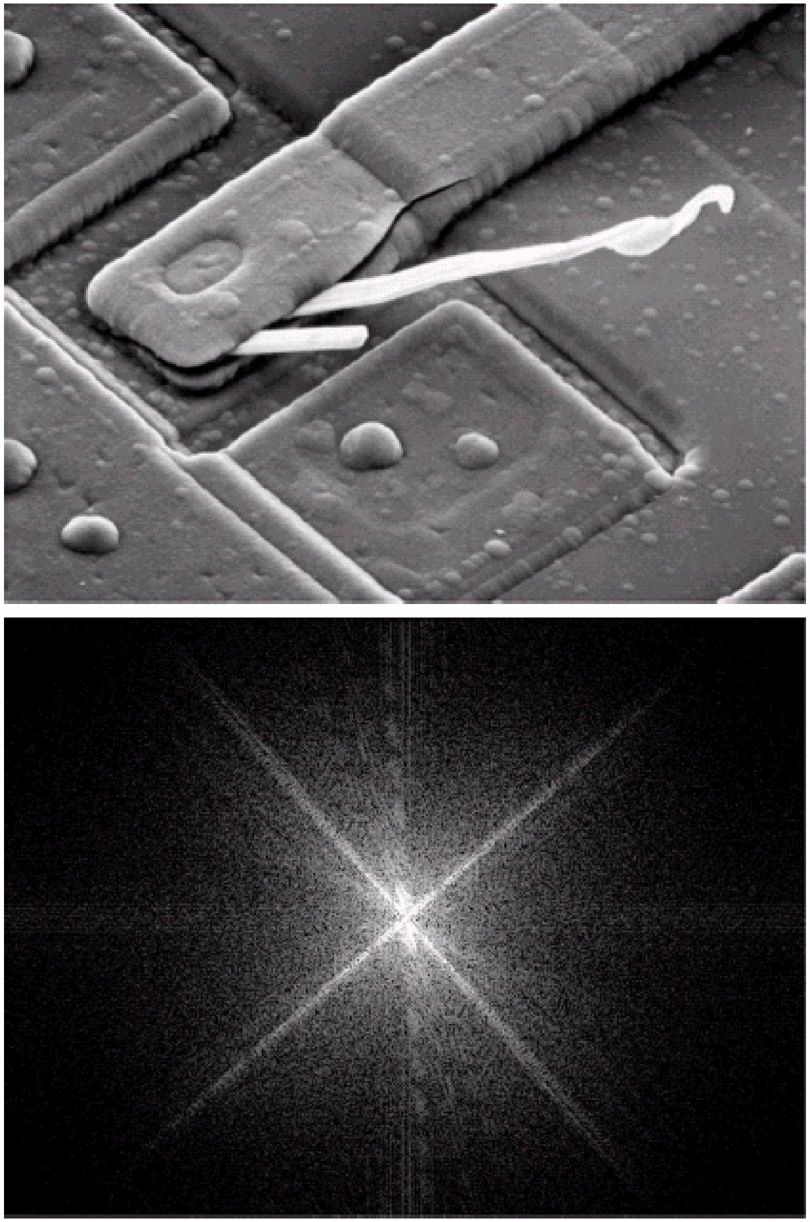


a b

FIGURE 4.3
(a) Image of a 20 × 40 white rectangle on a black background of size 512 × 512 pixels.
(b) Centered Fourier spectrum shown after application of the log transformation given in Eq. (3.2-2). Compare with Fig. 4.2.

Notice that the longer the time domain signal,
The shorter its Fourier transform

Example of 2-D DFT



a
b

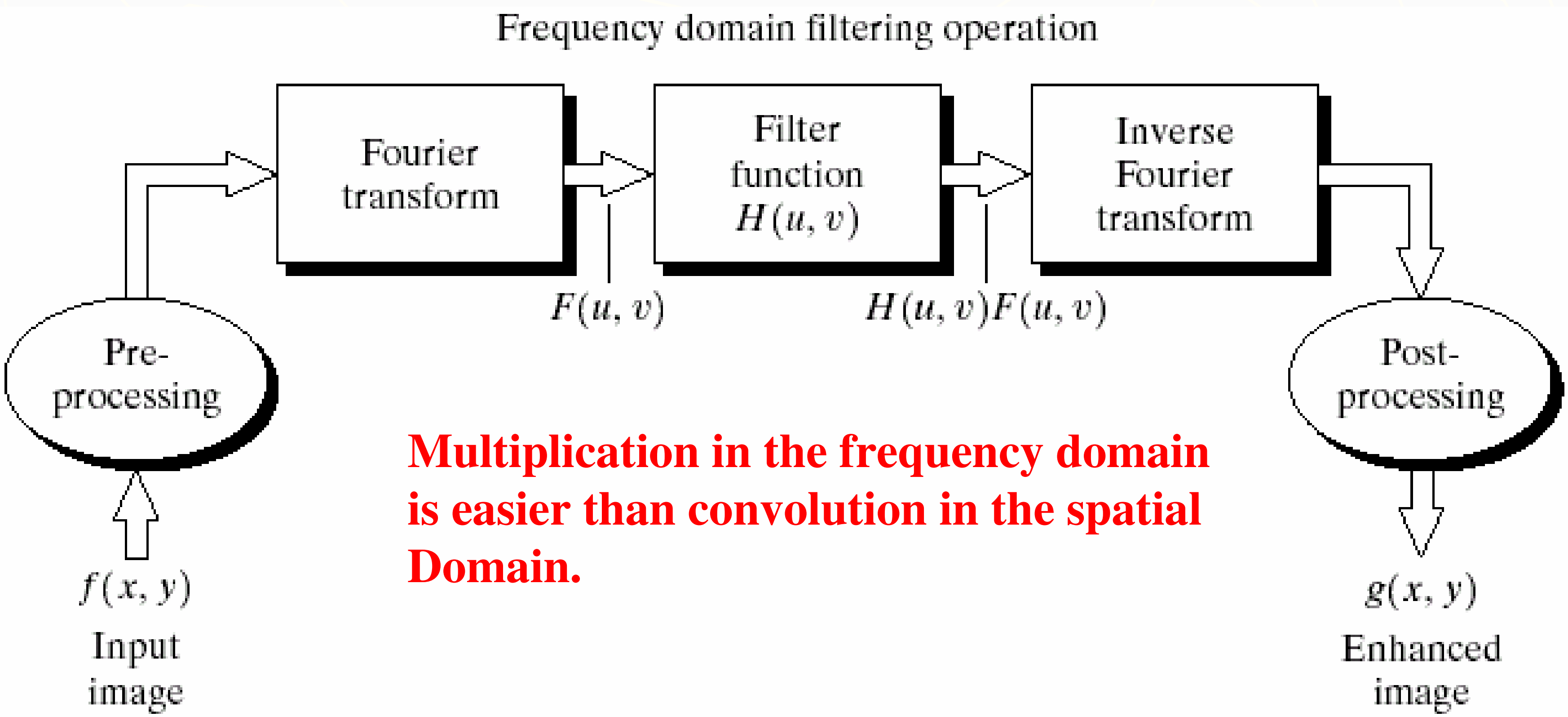
FIGURE 4.4
(a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Basic Concept of Filtering in the Frequency Domain

From Fourier Transform Property:

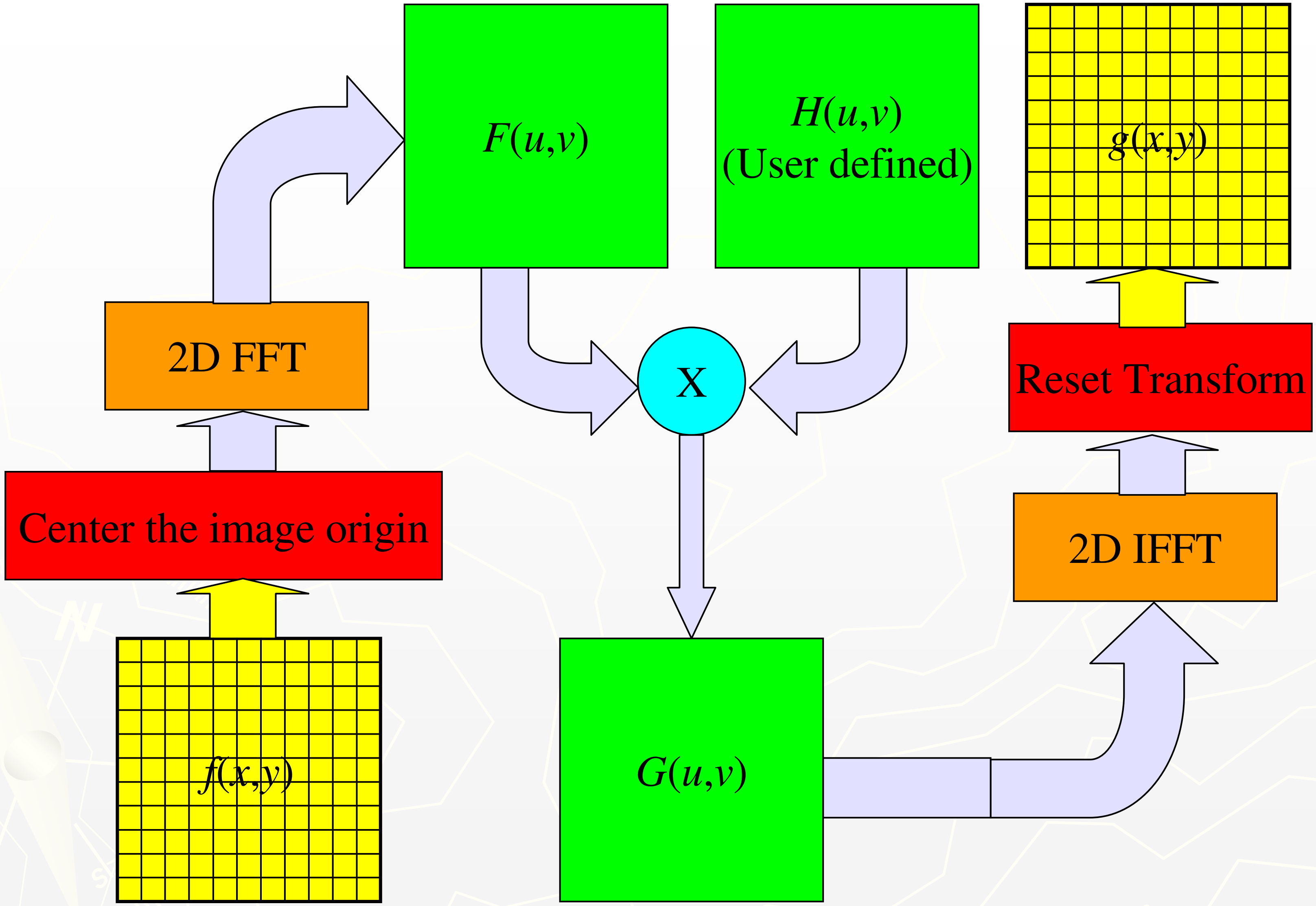
$$g(x, y) = f(x, y) * h(x, y) \Leftrightarrow F(u,v) \cdot H(u,v) = G(u,v)$$

We cam perform filtering process by using



Multiplication in the frequency domain is easier than convolution in the spatial Domain.

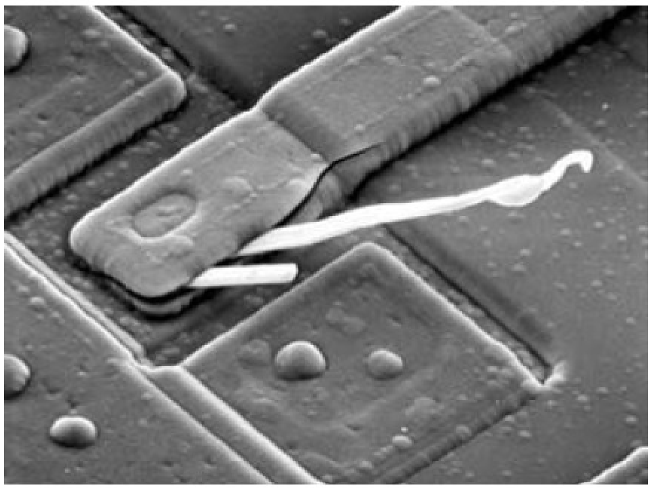
Filtering in the Frequency Domain with FFT shift

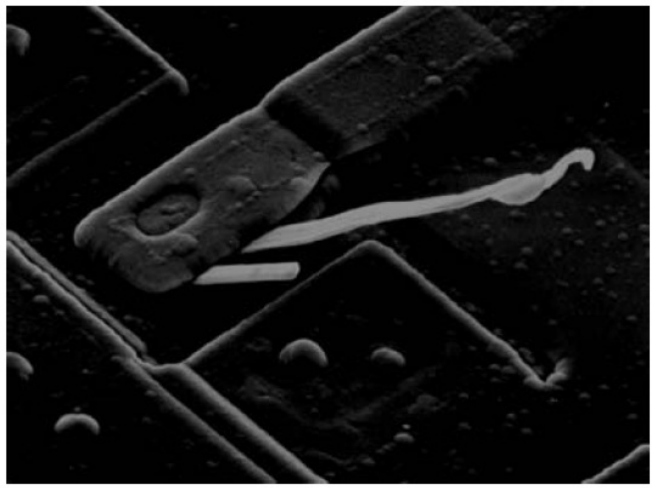


In this case, $F(u,v)$ and $H(u,v)$ must have the same size and have the zero frequency at the center.

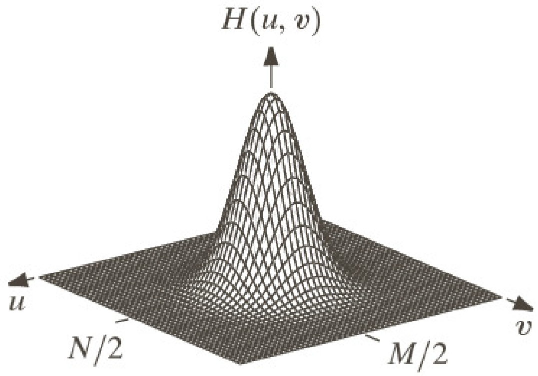
The Basic Filtering in the Frequency Domain

- In a filter H(u,v) that is 0 at the center of the transform and 1 elsewhere, what's the output image?





The Basic Filtering in the Frequency Domain



a b c
d e f

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

Zero-Phase-Shift Filters

$$g(x, y) = \Im^{-1}\{H(u,v)F(u,v)\}$$

$$F(u,v) = R(u,v) + jI(u,v)$$

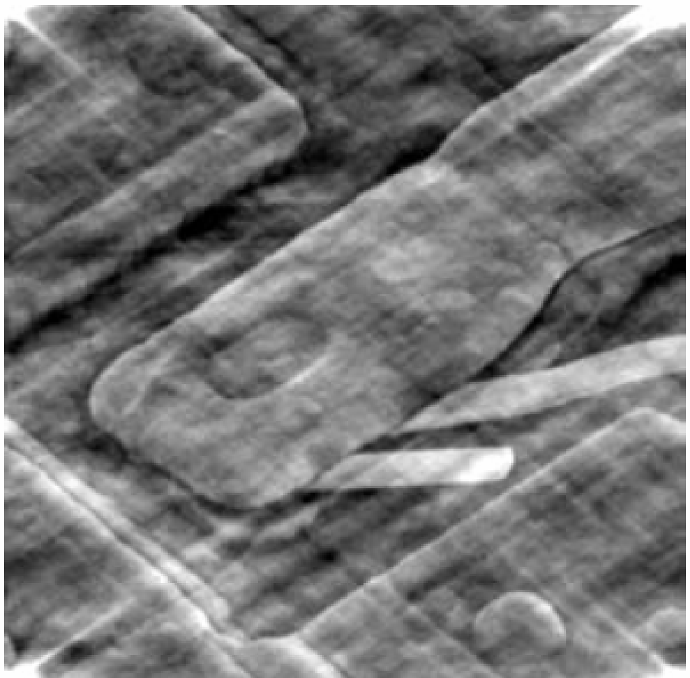
$$g(x, y) = \Im^{-1}\left[H(u,v)R(u,v) + jH(u,v)I(u,v)\right]$$

Filters affect the real and imaginary parts equally,

and thus no effect on the phase.

These filters are called **zero-phase-shift** filters

Examples: Nonzero-Phase-Shift Filters



a b

FIGURE 4.35
(a) Image resulting from multiplying by 0.5 the phase angle in Eq. (4.6-15) and then computing the IDFT. (b) The result of multiplying the phase by 0.25. The spectrum was not changed in either of the two cases.

Even s… s in the phase … ave drama… ndesirable) ef… filtered outpu…

Phase angle is multiplied by 0.5

Phase angle is multiplied by 0.25

2-D Convolution Theorem

1-D convolution

$$f(x)\star h(x) = \sum_{m=0}^{M-1} f(m)h(x-m)$$

2-D convolution

$$f(x,y)\star h(x,y) = \sum_{m=0}^{M-1}\sum_{n=0}^{N-1} f(m,n)h(x-m,y-n)$$

$$x = 0,1,2,\ldots,M-1;\ y = 0,1,2,\ldots,N-1.$$

$$f(x,y)\star h(x,y) \Leftrightarrow F(u,v)H(u,v)$$

$$f(x,y)h(x,y) \Leftrightarrow F(u,v)\star H(u,v)$$

Fourier Transform and Convolution

Fourier Transform Pairs

$$f(t) \star h(t) \Leftrightarrow H(\mu)F(\mu)$$

$$f(t)h(t) \Leftrightarrow H(\mu) \star F(\mu)$$

Correspondence Between Filtering in the Spatial and Frequency Domains (1)

Let H(u) denote the 1-D frequency domain Gaussian filter

$$H(u) = Ae^{-u^2/2\sigma^2}$$

The corresponding filter in the spatial domain

$$h(x) = \sqrt{2\pi}\sigma A e^{-2\pi^2\sigma^2 x^2}$$

1. Both components are Gaussian and real
2. The functions behave reciprocally

Correspondence Between Filtering in the Spatial and Frequency Domains (2)

Let $H(u)$ denote the difference of Gaussian filter

$$H(u) = Ae^{-u^2/2\sigma_1^2} - Be^{-u^2/2\sigma_2^2}$$

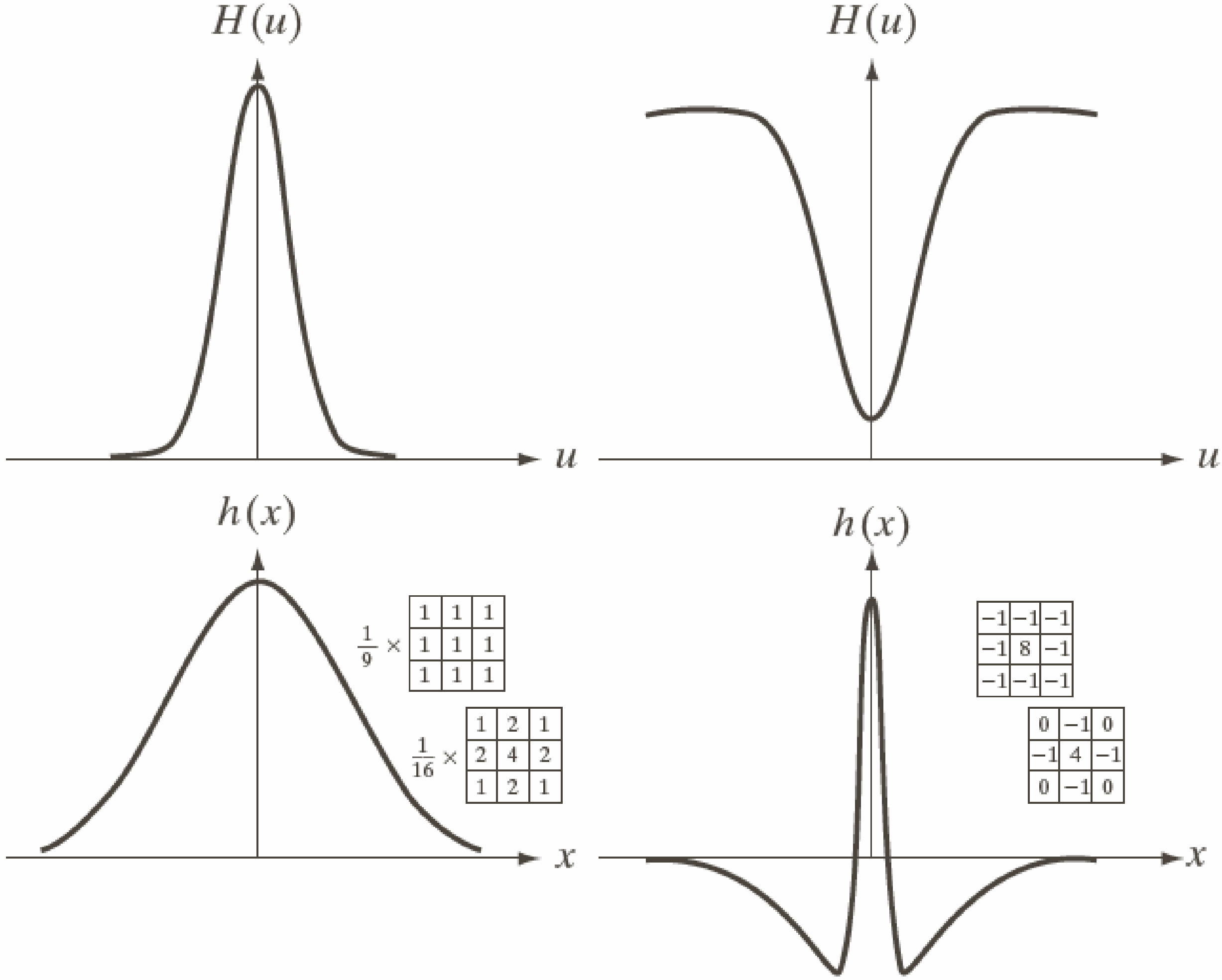
with $A \geq B$ and $\sigma_1 \geq \sigma_2$

The corresponding filter in the spatial domain

$$h(x) = \sqrt{2\pi}\sigma_1 Ae^{-2\pi^2\sigma_1^2x^2} - \sqrt{2\pi}\sigma_2 Ae^{-2\pi^2\sigma_2^2x^2}$$

High-pass filter or low-pass filter ?

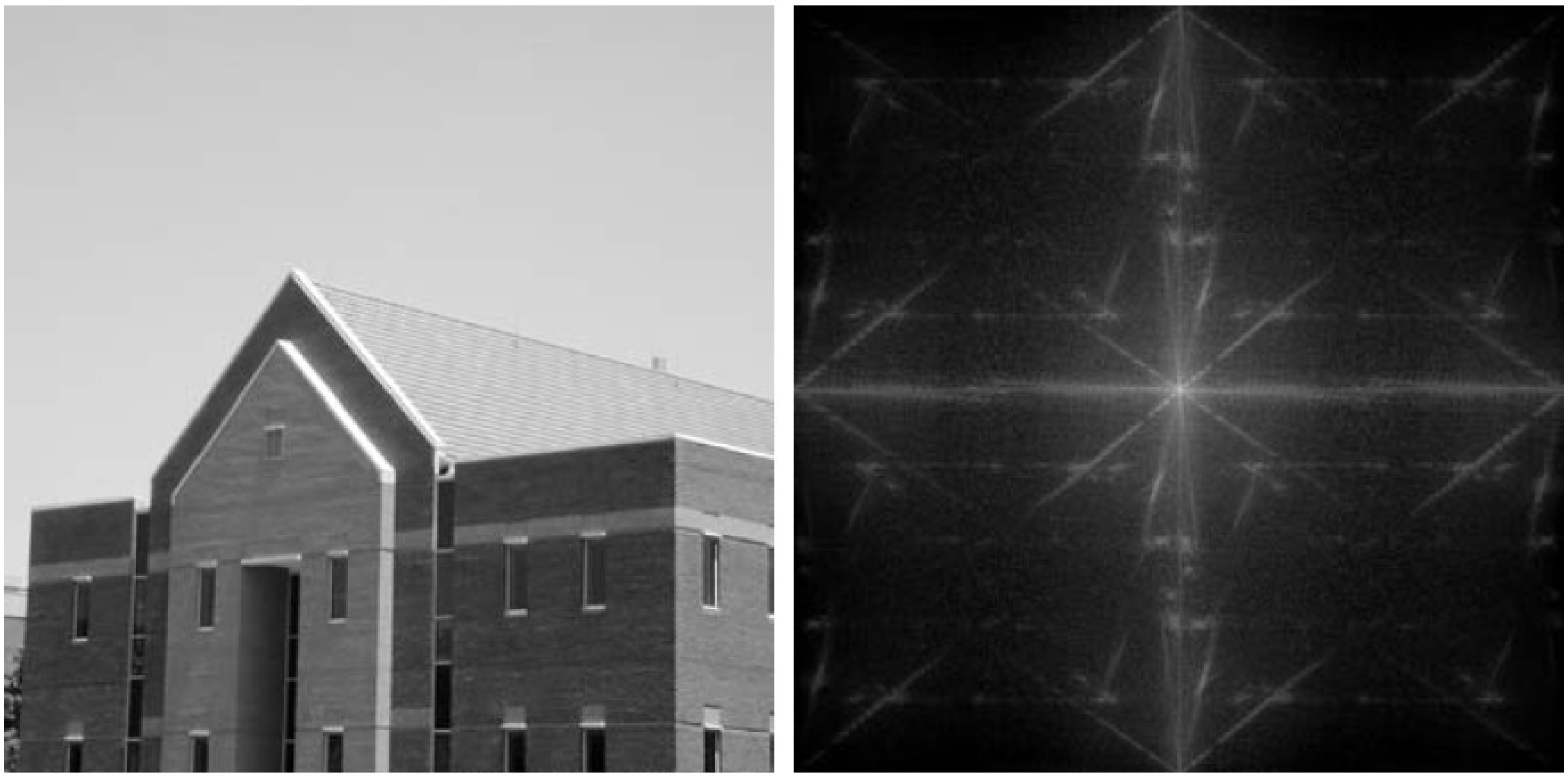
Correspondence Between Filtering in the Spatial and Frequency Domains (3)



a c
b d

FIGURE 4.37
(a) A 1-D Gaussian lowpass filter in the frequency domain. (b) Spatial lowpass filter corresponding to (a). (c) Gaussian highpass filter in the frequency domain. (d) Spatial highpass filter corresponding to (c). The small 2-D masks shown are spatial filters we used in Chapter 3.

Correspondence Between Filtering in the Spatial and Frequency Domains: Example

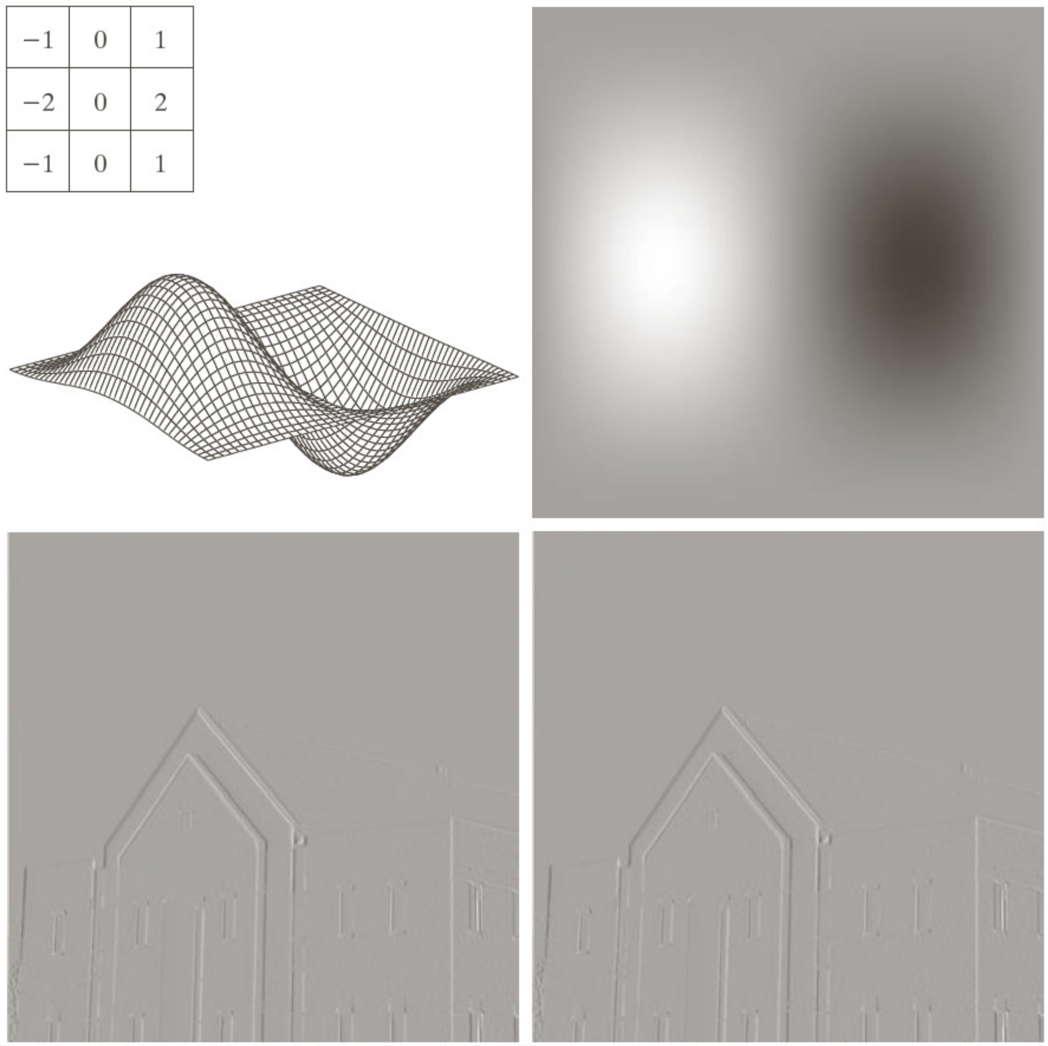


a b

FIGURE 4.38 (a) Image of a building, and (b) its spectrum.

600x600

Correspondence Between Filtering in the Spatial and Frequency Domains: Example



a b
c d

FIGURE 4.39
(a) A spatial mask and perspective plot of its corresponding frequency domain filter. (b) Filter shown as an image. (c) Result of filtering Fig. 4.38(a) in the frequency domain with the filter in (b). (d) Result of filtering the same image with the spatial filter in (a). The results are identical.

Image Smoothing Using Filter Domain Filters: ILPF

Ideal Lowpass Filters (ILPF)

$$H(u,v) = \begin{cases} 1 & \text{if } D(u,v) \le D_0 \\ 0 & \text{if } D(u,v) > D_0 \end{cases}$$

D_0 is a positive constant and $D(u,v)$ is the distance between a point (u,v) in the frequency domain and the center of the frequency rectangle

$$D(u,v) = \left[(u - P/2)^2 + (v - Q/2)^2 \right]^{1/2}$$

Ideal Lowpass Filter

Ideal LPF Filter Transfer function

$$H(u,v) = \begin{cases} 1 & D(u,v) \le D_0 \\ 0 & D(u,v) > D_0 \end{cases}$$

where $D(u,v)$ = Distance from (u,v) to the center of the mask.

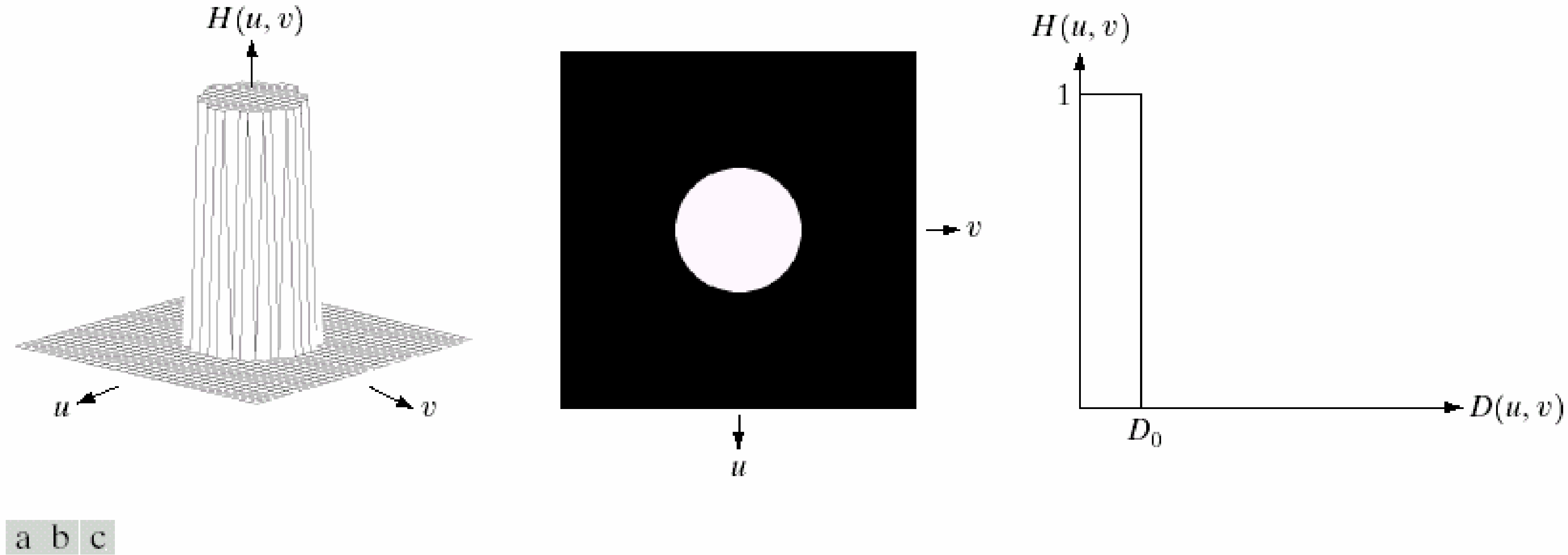
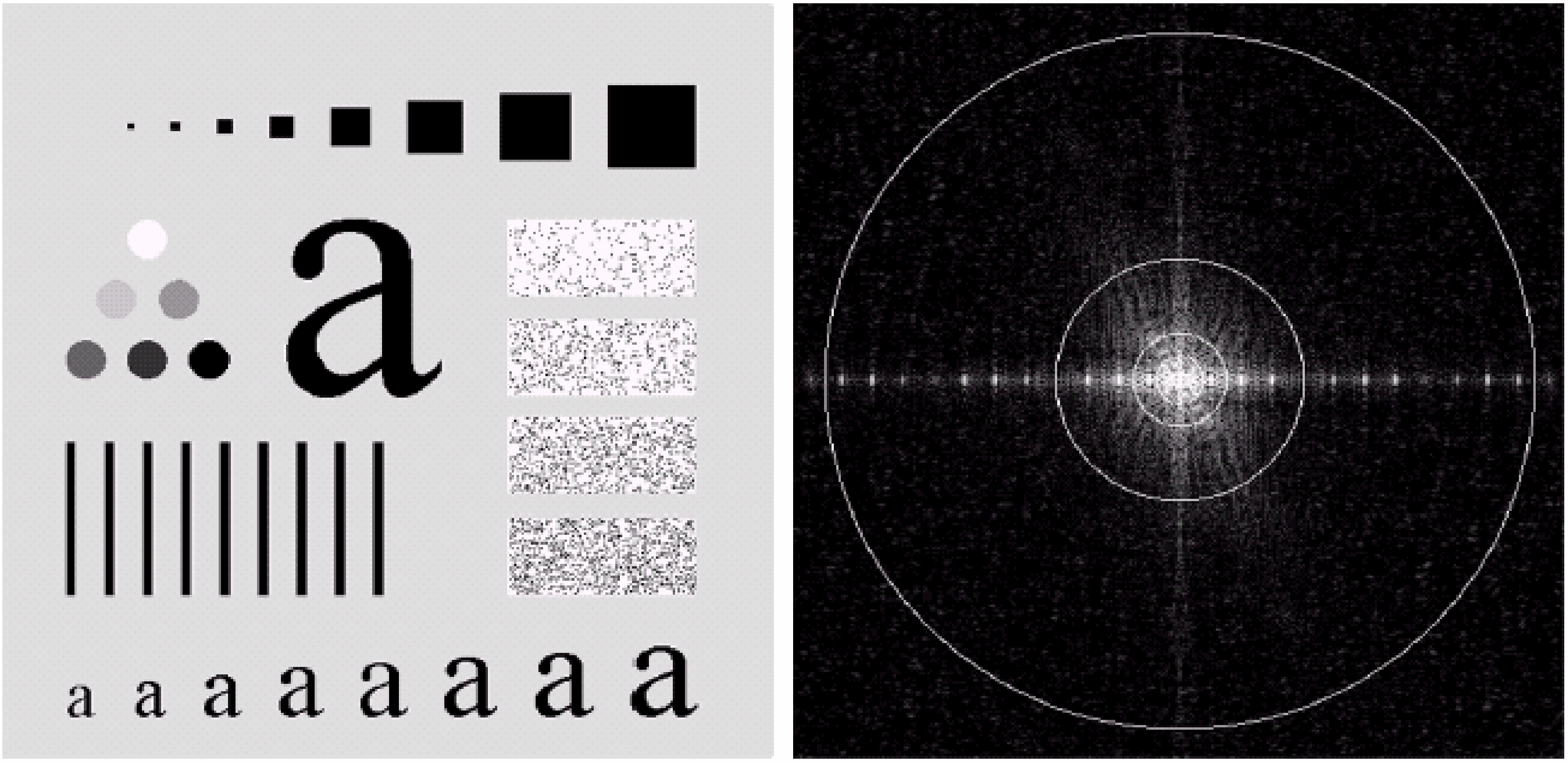


FIGURE 4.10 (a) Perspective plot of an ideal lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

Examples of Ideal Lowpass Filters

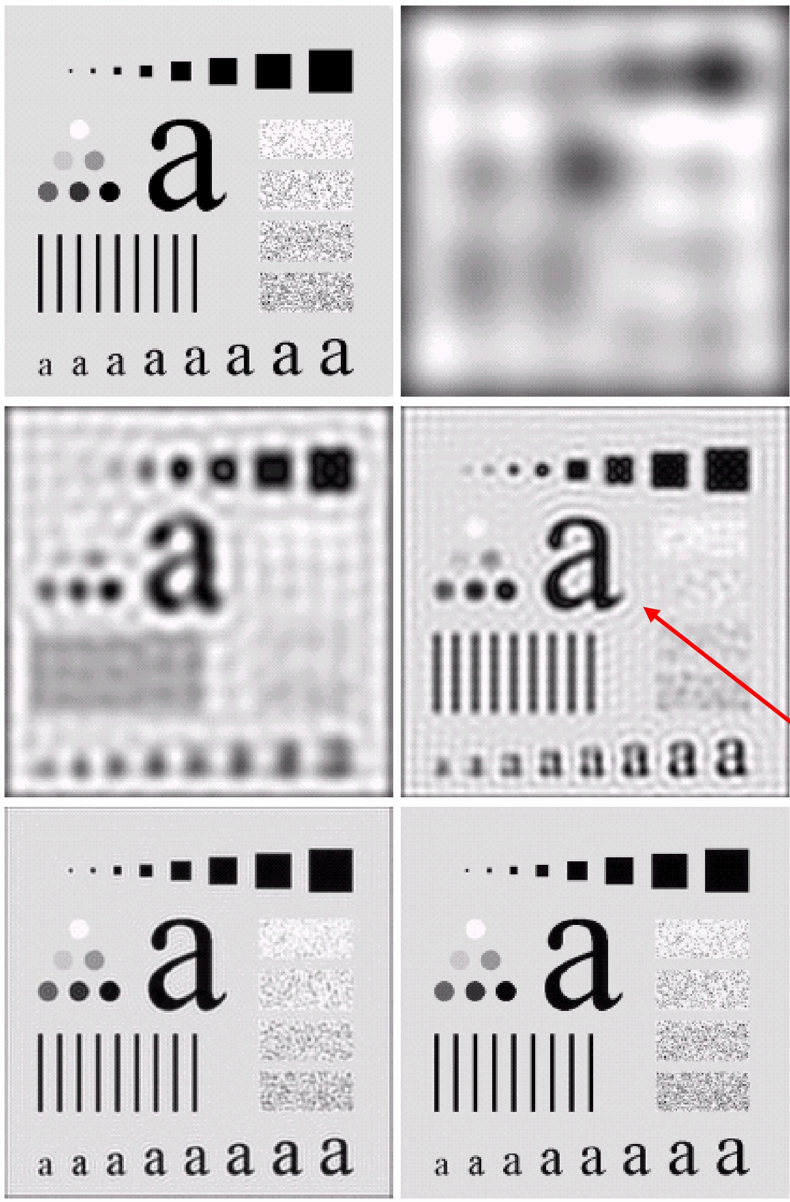


a b

FIGURE 4.11 (a) An image of size 500 × 500 pixels and (b) its Fourier spectrum. The superimposed circles have radii values of 5, 15, 30, 80, and 230, which enclose 92.0, 94.6, 96.4, 98.0, and 99.5% of the image power, respectively.

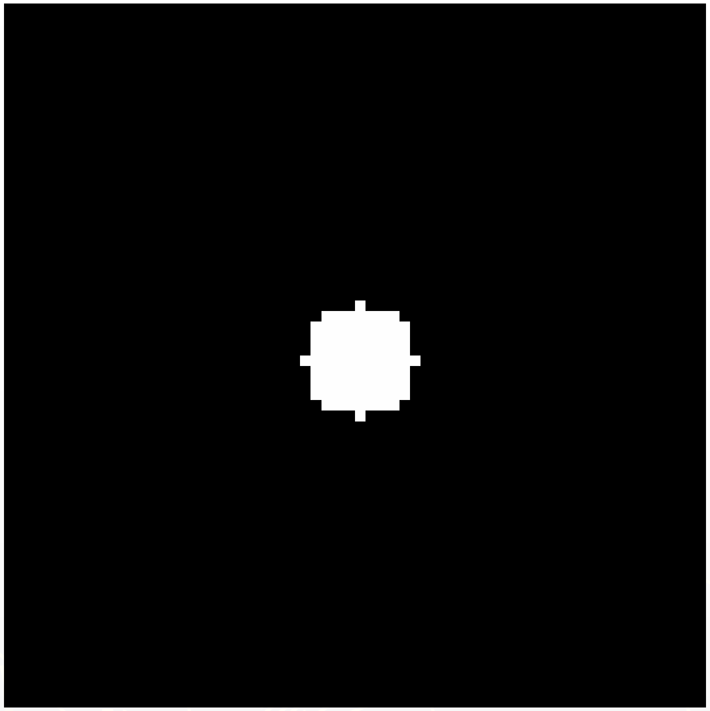
The smaller D_0, the more high frequency components are removed.

Results of Ideal Lowpass Filters



Ringing effect can be obviously seen!

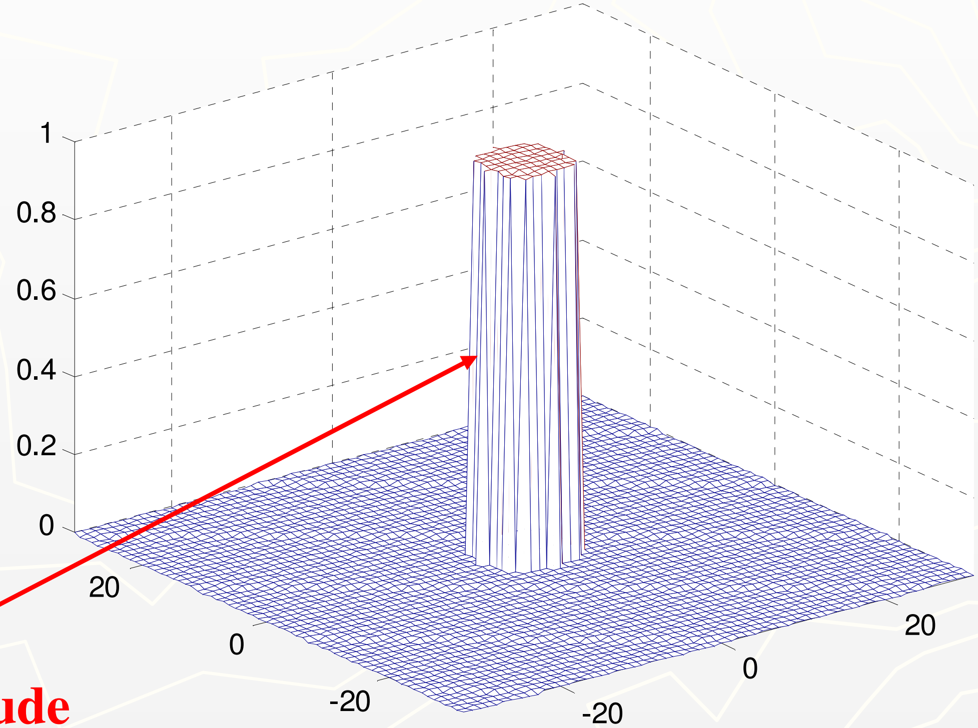
How ringing effect happens



Ideal Lowpass Filter
with $D_0 = 5$

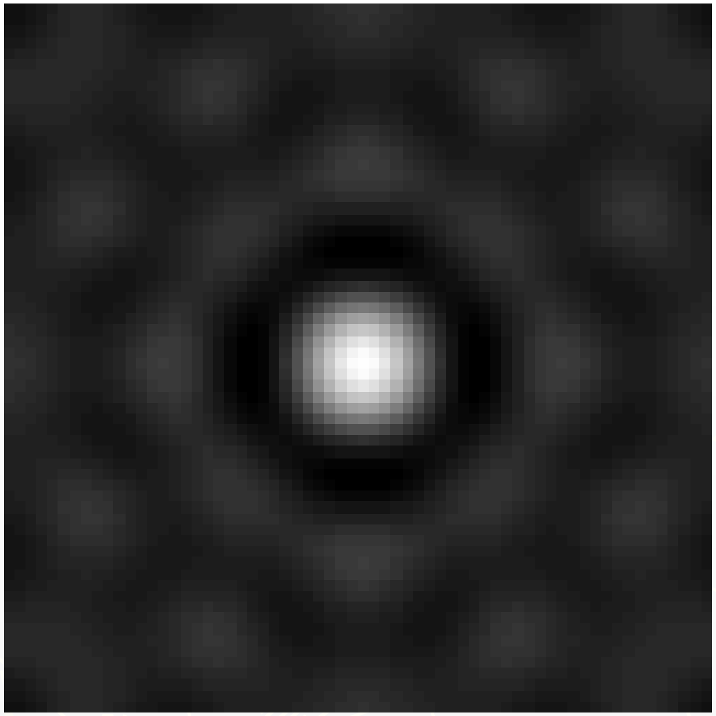
$$H(u,v) = \begin{cases} 1 & D(u,v) \le D_0 \\ 0 & D(u,v) > D_0 \end{cases}$$

Surface Plot



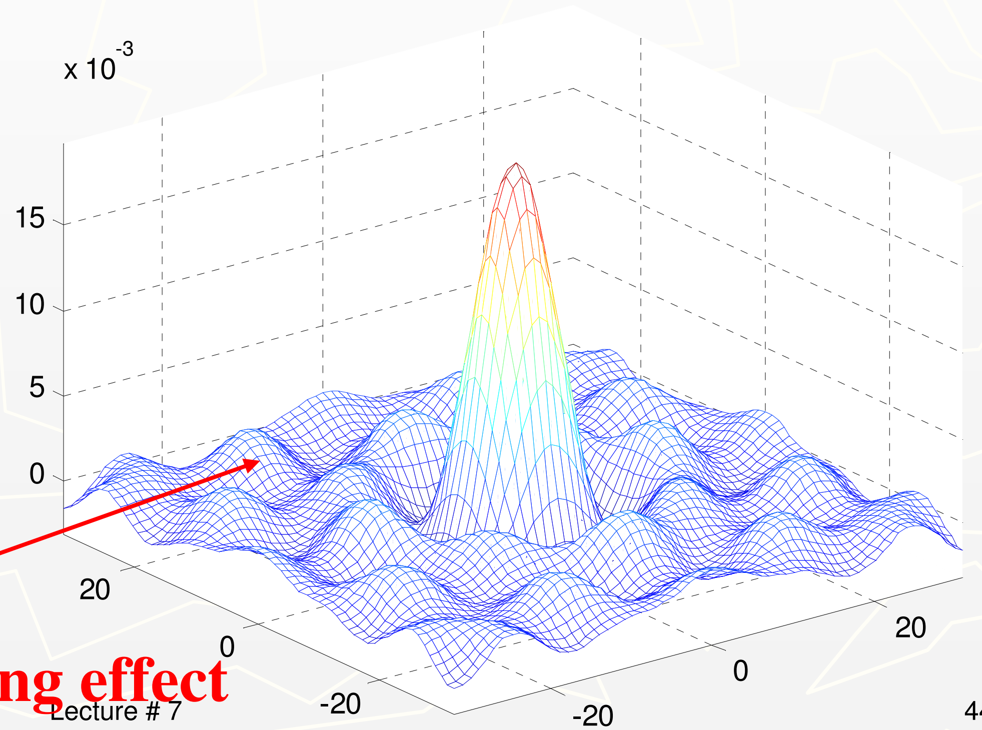
Abrupt change in the amplitude

How ringing effect happens (cont.)



Spatial Response of Ideal Lowpass Filter with $D_0 = 5$

Surface Plot



Ripples that cause ringing effect

How ringing effect happens (cont.)

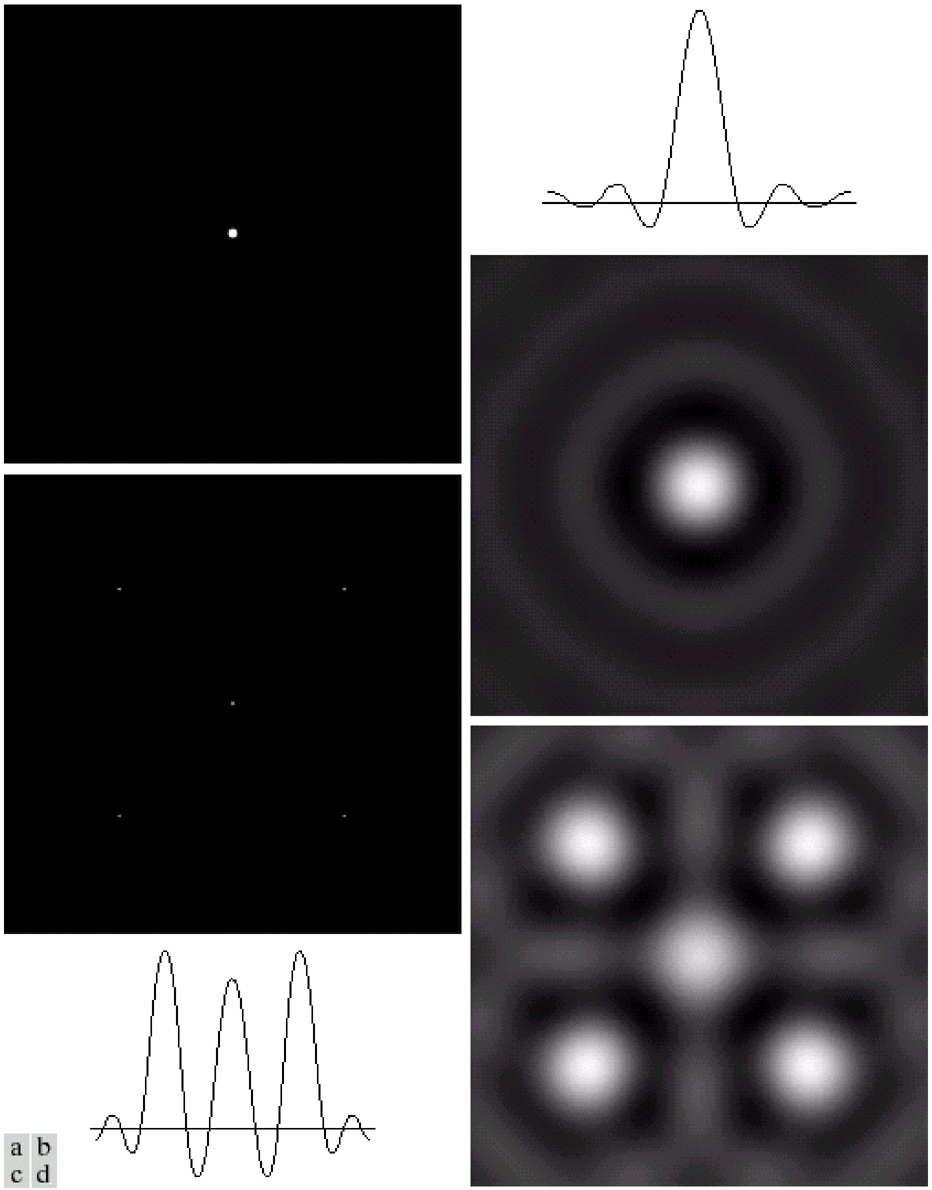


FIGURE 4.13 (a) A frequency-domain ILPF of radius 5. (b) Corresponding spatial filter (note the ringing). (c) Five impulses in the spatial domain, simulating the values of five pixels. (d) Convolution of (b) and (c) in the spatial domain.

Butterworth Lowpass Filter

Transfer function

$$H(u,v) = \frac{1}{1 + [D(u,v)/D_0]^{2N}}$$

Where D_0 = Cut off frequency, N = filter order.

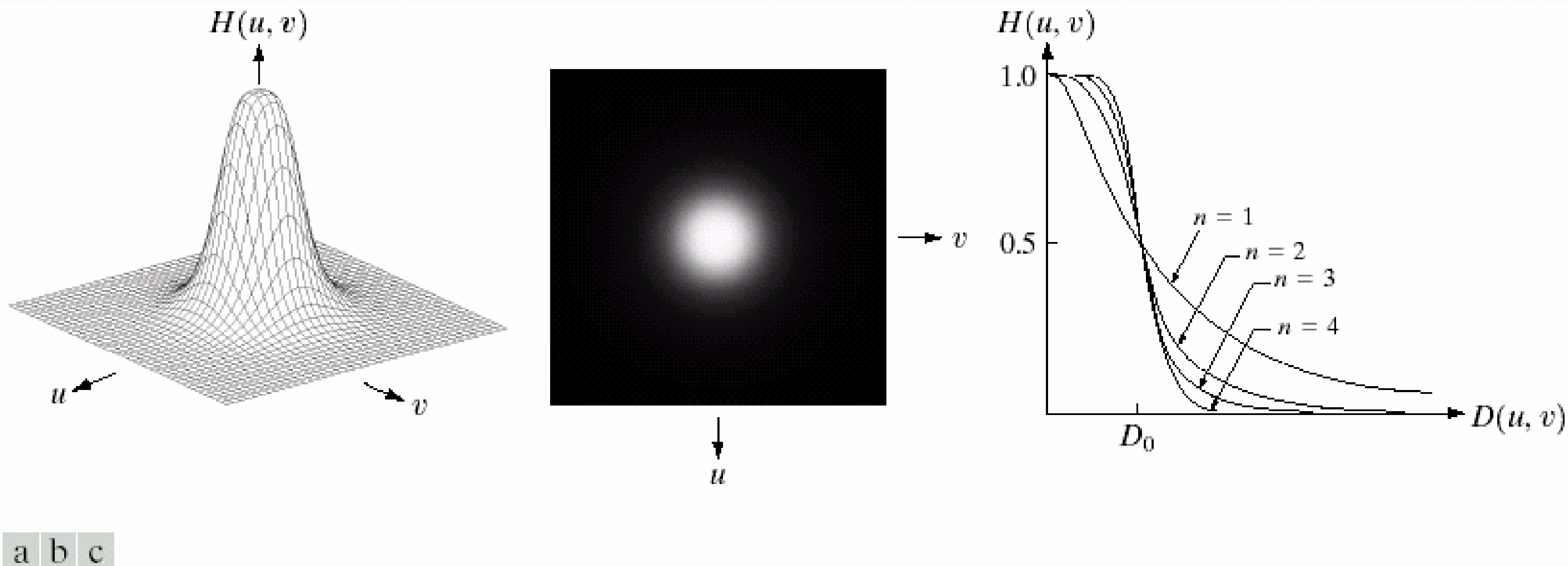


FIGURE 4.14 (a) Perspective plot of a Butterworth lowpass filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Results of Butterworth Lowpass Filters

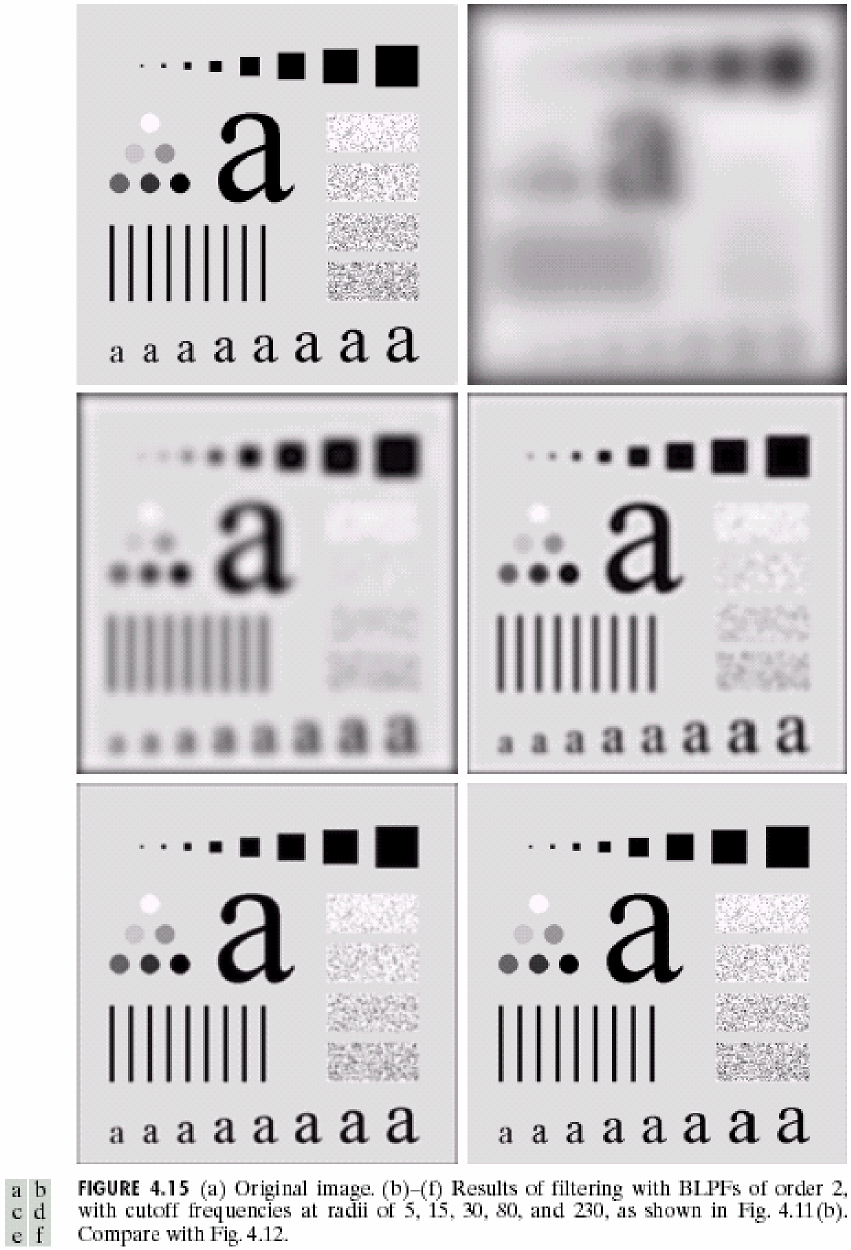


FIGURE 4.15 (a) Original image. (b)–(f) Results of filtering with BLPFs of order 2, with cutoff frequencies at radii of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Fig. 4.12.

There is less ringing effect compared to those of ideal lowpass filters!

Spatial Masks of the Butterworth Lowpass Filters

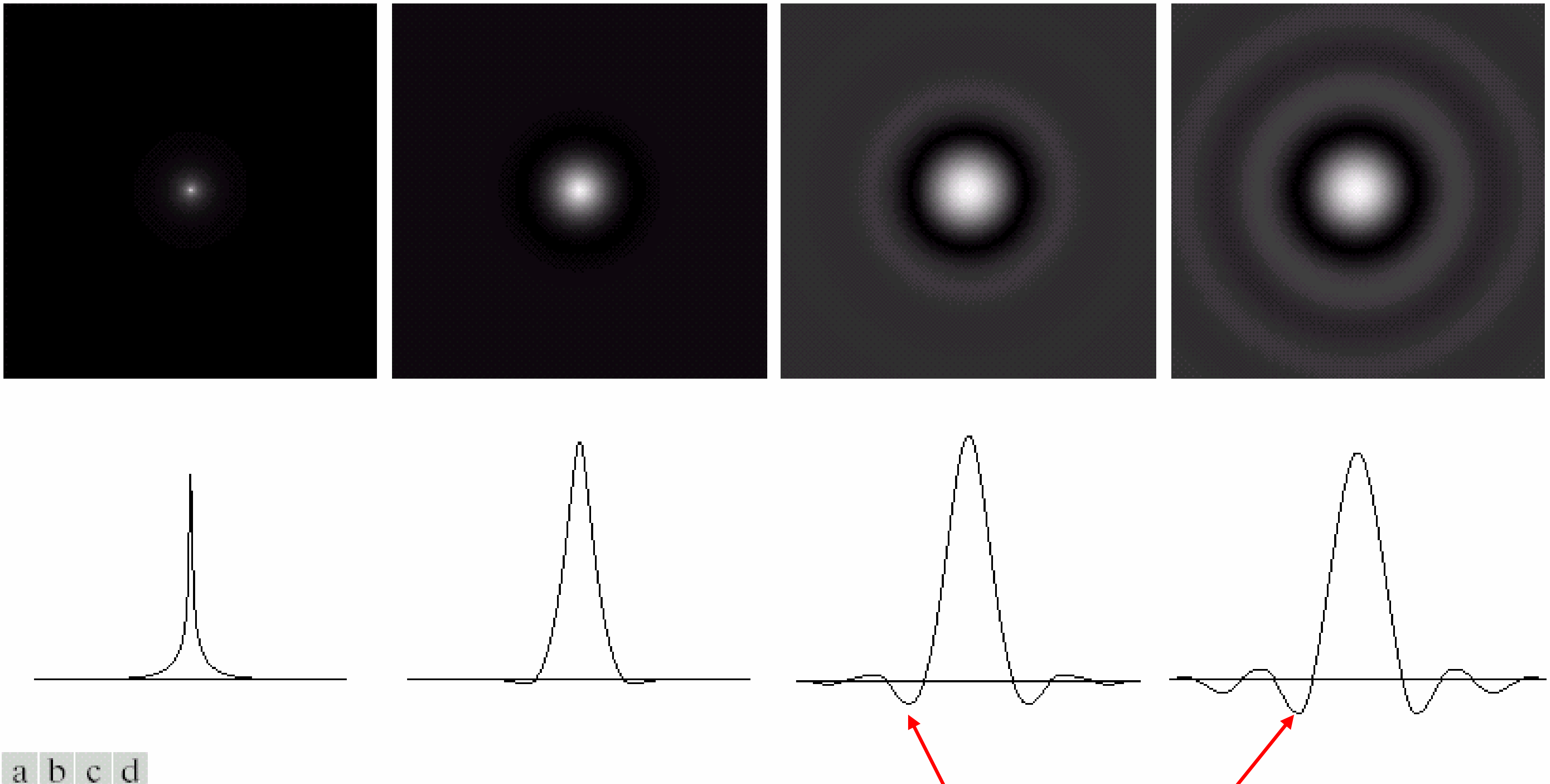


FIGURE 4.16 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding gray-level profiles through the center of the filters (all filters have a cutoff frequency of 5). Note that ringing increases as a function of filter order.

Some ripples can be seen.

Gaussian Lowpass Filter

Transfer function

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$

Where D_0 = spread factor.

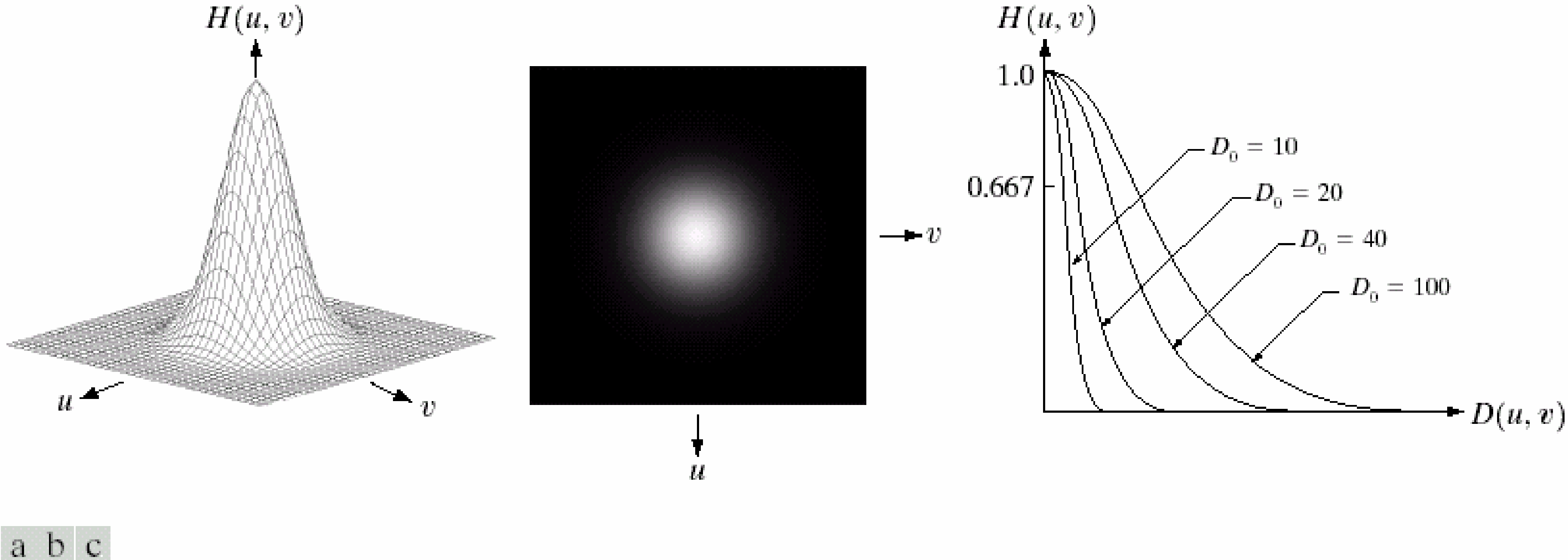
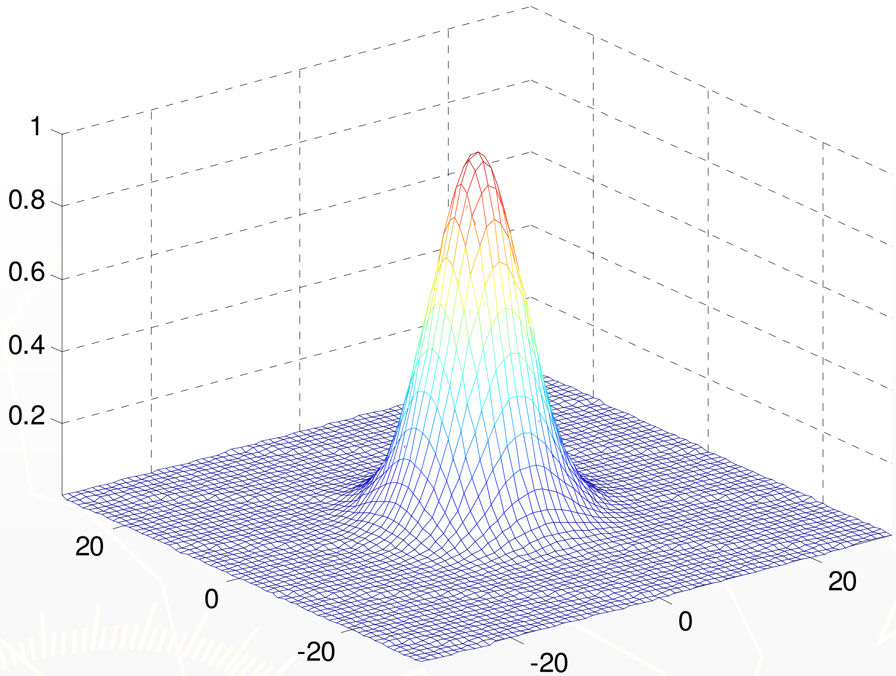


FIGURE 4.17 (a) Perspective plot of a GLPF transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections for various values of D_0.

Note: the Gaussian filter is the only filter that has no ripple and hence no ringing effect.

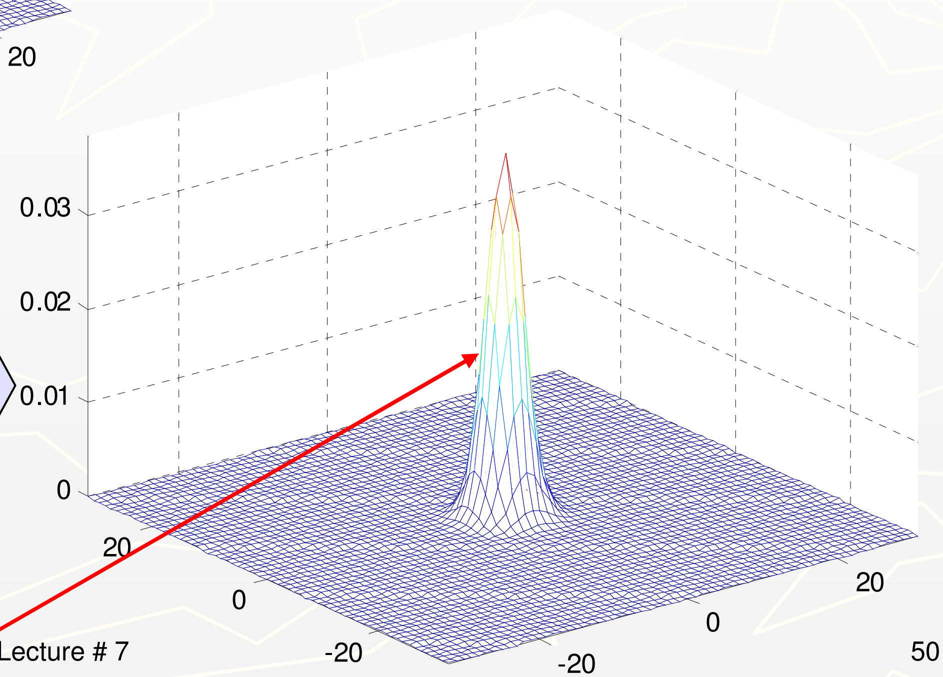
Gaussian Lowpass Filter (cont.)

$$H(u,v) = e^{-D^2(u,v)/2D_0^2}$$



Gaussian lowpass filter with $D_0 = 5$

Spatial respones of the Gaussian lowpass filter with $D_0 = 5$



Gaussian shape

Results of Gaussian Lowpass Filters

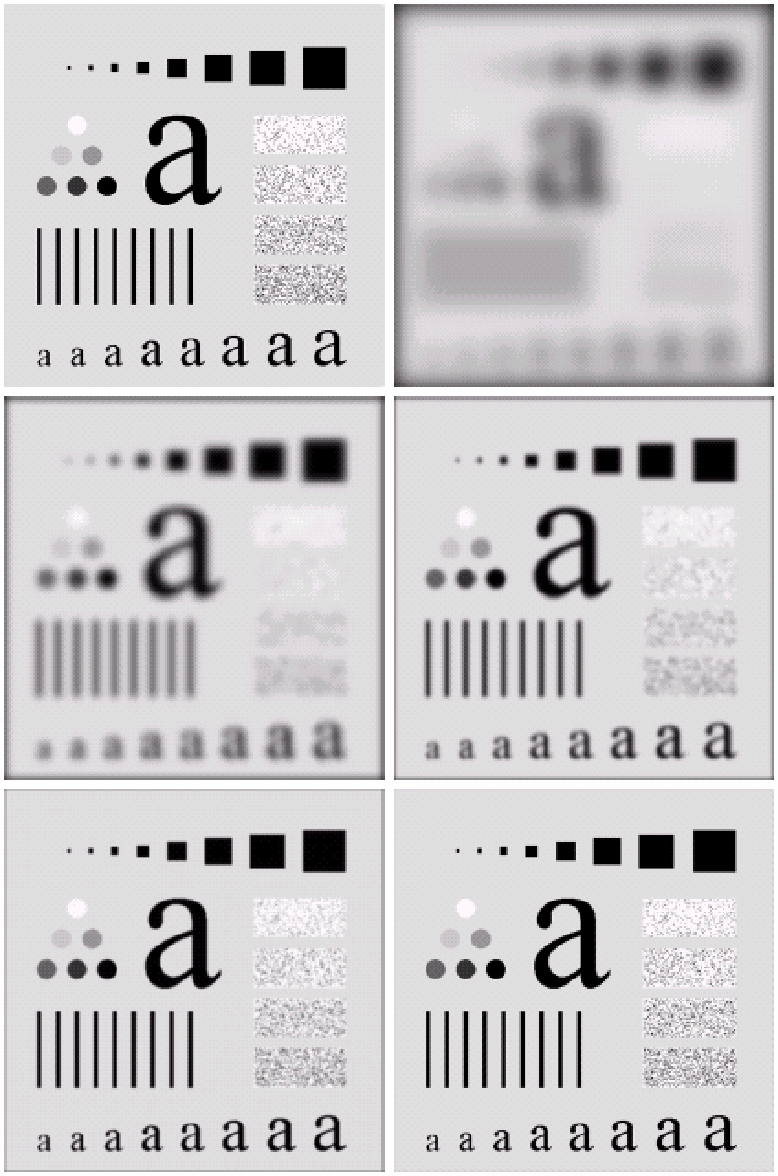


FIGURE 4.18 (a) Original image. (b)–(f) Results of filtering with Gaussian lowpass filters with cutoff frequencies set at radii values of 5, 15, 30, 80, and 230, as shown in Fig. 4.11(b). Compare with Figs. 4.12 and 4.15.

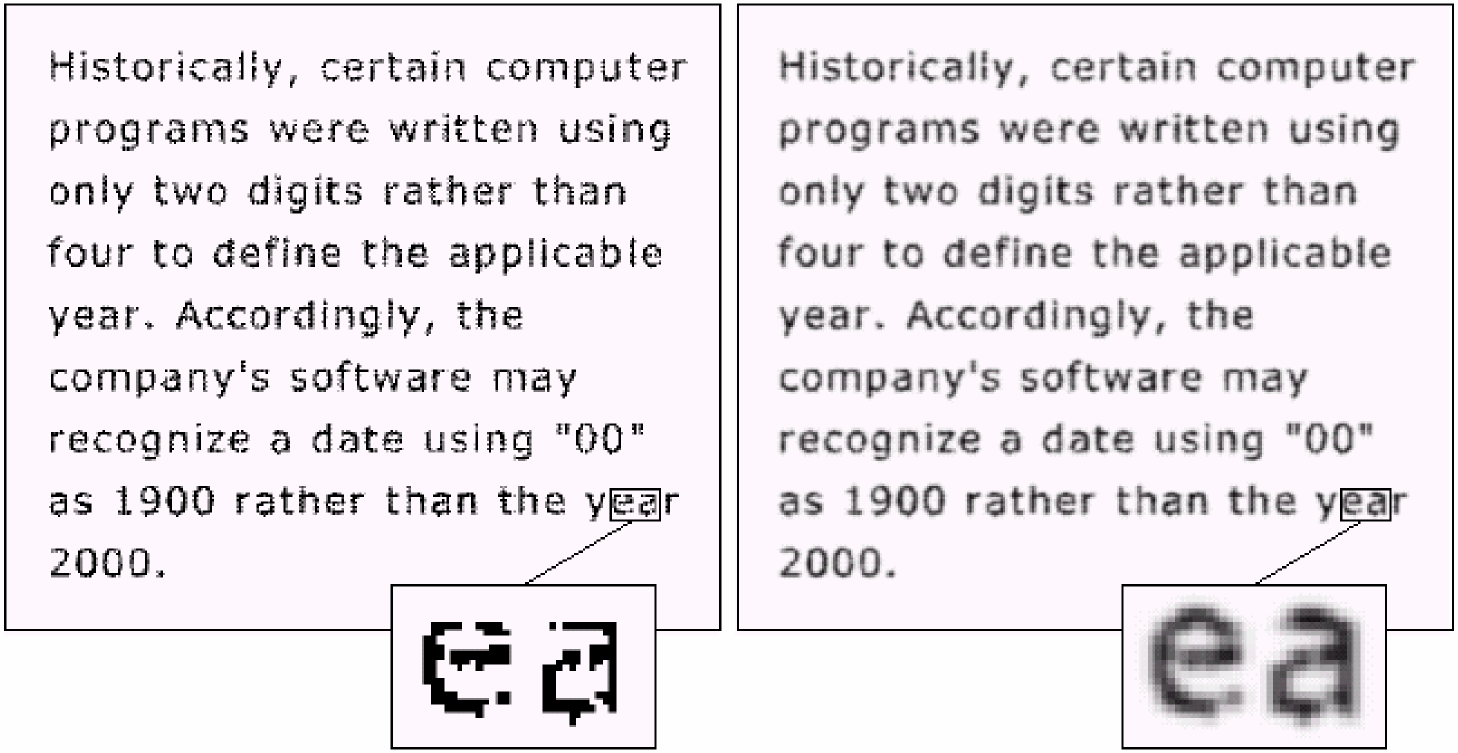
a b
c d
e f

No ringing effect!

Application of Gaussian Lowpass Filters

a b

FIGURE 4.19
(a) Sample text of poor resolution (note broken characters in magnified view).
(b) Result of filtering with a GLPF (broken character segments were joined).



Original image

Better Looking

The GLPF can be used to remove jagged edges and "repair" broken characters.

Application of Gaussian Lowpass Filters (cont.)

a b c

FIGURE 4.20 (a) Original image (1028 × 732 pixels). (b) Result of filtering with a GLPF with $D_0 = 100$. (c) Result of filtering with a GLPF with $D_0 = 80$. Note reduction in skin fine lines in the magnified sections of (b) and (c).